Optimal Sterile-Insect and Pheromone Deployment for False Codling Moth Suppression: A Stability- and Sensitivity-Driven Model

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Abstract

A nonlinear host–pest model was formulated to assess combined sterile-insect technique and pheromone trapping against the false-codling moth *Thaumatotibia leucotreta*. Next-generation methods yielded the basic reproduction number R_0, which equalled 0.86 under baseline field estimates. Parameter sensitivity showed that a 15% increase in the sterile-mating rate λ_4 cut R_0 by 48%. Centre-manifold reduction uncovered a backward bifurcation at $R_0 = 1$; therefore, eradication requires both $R_0 < 1$ and releases large enough to surmount the fold. Pontryagin's minimum principle supplied an optimal weekly schedule: 2 000 sterile males ha^{-1} together with 25 pheromone traps ha^{-1}. This control mix lowered peak larval density by 38% and accelerated host-plant recovery, raising biomass by 12% within 60 d relative to constant-rate programmes. All biological parameters were sourced from peer-reviewed laboratory and field studies; no

proprietary data were used. The results endorse integrated deployment of sterile males and pheromone lures as a cost-effective strategy for protecting high-value perennial crops.

Keywords— Mathematical Modeling, Crop pest model, False Codling Moth, Epidemiology Model, Pheromones, sterile Insect

1 Introduction

Integrated pest management (IPM) has evolved into a crucial framework for sustainable agriculture, aiming to balance effective pest suppression with environmental and economic considerations. Mathematical models have become indispensable in IPM by enabling quantitative exploration of complex interactions between pests, hosts, and control measures. Such models guide decision-making without resorting immediately to costly or ecologically risky field trials. In particular, host–pest interaction models formulated as systems of ordinary differential equations (ODEs) allow rigorous stability analysis, threshold determination through reproduction numbers, and optimal-control design via the Pontryagin maximum principle. Foundational contributions include the next-generation matrix method for threshold analysis [23], age-structured population frameworks [5, 20], and optimal-control formulations in pest dynamics [18]. The proliferation of computational tools has further facilitated sensitivity analyses such as Latin Hypercube Sampling and partial rank correlation coefficients to identify key parameters driving system behavior [7]. These mathematical advances underpin modern IPM strategies by clarifying when and how control tactics must be applied to achieve desired reductions in pest density.

The false codling moth, *Thaumatotibia leucotreta* (Meyrick) (Lepidoptera: Tortricidae), poses a severe threat to sub-Saharan African horticulture. Its larvae bore into fruits of citrus, avocado, cotton, maize, and stone fruit, often escaping contact insecticides by feeding internally [8, 16] . Under favorable warm and humid conditions, development proceeds through up to five generations per year, while temperatures below 10 °C halt growth [10, 21]. These biological traits render *T. leucotreta* a high-priority target for area-wide suppression efforts.

Two environmentally benign tactics have gained prominence against polyphagous pests like the false codling moth: the sterile insect technique (SIT) and pheromone trapping. SIT involves mass-rearing, sterilization (via irradiation or chemosterilants), and systematic release of sterile males to mate with wild females, thereby reducing viable offspring and causing population collapse over successive generations [17, 12, 15]. Pheromone traps exploit species-specific sex pheromones commonly blends of (Z)-8-dodecenyl acetate and

(E)-8-dodecenyl acetate for *T. leucotreta* to lure and capture males, providing real-time monitoring and population suppression [13, 19]. Experimental studies have documented the independent efficacy of each tactic [4**?**], yet field trials combining SIT and pheromone traps remain scarce.

Mathematical models have captured individual effects: age-structured pheromone trap models demonstrated threshold trap densities for suppression [9], while classical SIT models quantified release rates required for eradication [2, 6]. However, few studies have integrated both control methods within a unified framework for T. leucotreta, and sensitivity analyses highlighting the most impactful parameters remain limited. Similarly, optimal-control investigations have primarily considered single tactics, neglecting potential synergies and cost trade-offs when tactics are co-implemented [18, 24].

Addressing these gaps, the present study formulates a ten-compartment ODE model coupling host plant growth with egg, larval, pupal, and adult stages of both wild and sterile male moths, along with pheromone trap dynamics. The model incorporates Holling-type II functional responses for larval feeding and mass-action mating interactions, reflecting key biological processes. Analytical contributions include proofs of positivity, boundedness, and existence of equilibria; derivation of the basic reproduction number R_0 via the next-generation matrix [23]; and identification of backward bifurcation through centre-manifold techniques [11]. A comprehensive sensitivity analysis ranks parameters by their influence on R_0, guiding targeted control efforts. Optimal-control theory is applied to determine time-varying release rates and pheromone densities that minimize pest density and control costs under realistic resource constraints.

The study aims to (1) characterize system thresholds and stability of pest-free and coexistence equilibria, (2) quantify parameter sensitivities to reveal high-leverage intervention points, and (3) design cost-effective combined control strategies that leverage SIT–pheromone synergies. Results will inform guidelines for integrated application of sterile insects and pheromone traps against *T. leucotreta*, enhancing IPM protocols in perennial crop systems.

2 Model Formulation

2.1 State variables, controls

We track ten state variables and two controls as functions of time t:

Symbol	Description (all functions of t)
Q	Susceptible host biomass (fruit or shoot mass)
E	Viable eggs laid by fertilised females
L	Feeding larvae
P	Pupae
F	Fertile females
M	Fertile males
Z	Fertilised females (egg-laying)
V	Non-fertilised females that mated with steriles
S	Sterile males present in the orchard
P_{trap}	Active pheromone lure (equivalent female units)
u_1	Release rate of sterile males
u_2	Replacement rate of pheromone lures

Table 1: State and control variables.

2.2 Larval Feeding and Mating Responses

We use two standard kernels:

Larval feeding: Consumption of host tissue by larvae saturates because individual larvae are limited by gut capacity and entrance holes. A Holling type-II term [14, 3]

$$\frac{\xi Q L}{m + Q}$$

captures this decelerating intake, where ξ is the maximum attack rate and m the half-saturation constant. The same structure is widely used in fruit–borer systems [1] Adult mating. Encounters between males and females occur in open air and are not subject to handling time; sterile and fertile males compete in proportion to their densities. Hence simple mass action

$$\frac{M}{M+S}\,F, \qquad \frac{S}{M+S}\,F$$

is appropriate and preserves the standard SIT framework [4].

2.3 Model Flow Chart

2.4 Governing Equations

Let the controls $u_1(t)$ and $u_2(t)$ be Lebesgue-integrable on $[0, T]$ and bounded above by operational maxima $u_1^{\max}$ and $u_2^{\max}$. The deterministic model reads

$$
\begin{aligned}
\dot{Q} &= \alpha Q\Big(1 - \frac{Q}{C}\Big) - \frac{\xi Q L}{m + Q} - \mu_h Q, \\
\dot{E} &= r Z\Big(1 - \frac{E}{A}\Big) - (\lambda_1 + \mu_E) E, \\
\dot{L} &= \lambda_1 E + \frac{a \xi Q L}{m + Q} - (\lambda_2 + \mu_L) L, \\
\dot{P} &= \lambda_2 L - (\lambda_3 + \mu_P) P, \\
\dot{F} &= \kappa \lambda_3 P + \delta_1 Z + \delta_2 V - \left[\lambda_4 \frac{M}{M + S} + \mu_F\right] F, \\
\dot{M} &= (1 - \kappa)\lambda_3 P - \left[\epsilon_1 \frac{P_{\text{trap}}}{F + P_{\text{trap}}} + \mu_M\right] M, \\
\dot{Z} &= \lambda_4 \frac{M}{M + S} F - (\delta_1 + \mu_Z) Z, \\
\dot{V} &= \lambda_4 \frac{S}{M + S} F - (\delta_2 + \mu_V) V, \\
\dot{S} &= u_1(t) - \left[\epsilon_2 \frac{P_{\text{trap}}}{F + P_{\text{trap}}} + \mu_S\right] S, \\
\dot{P}_{\text{trap}} &= u_2(t) - \nu P_{\text{trap}}.
\end{aligned}
\tag{1}
$$

All parameters are positive constants; their baseline values and sources are listed in Table 2. Controls enter linearly, which simplifies the Pontryagin optimality system.

2.5 Non-dimensional scaling

We introduce scaled time and states to ease numerical work:

$$
\tilde{t} = \lambda_3\, t, \quad \tilde{Q} = Q/C, \quad \tilde{L} = L/L_0, \quad \tilde{u}_1 = u_1/u_1^{\max}, \quad \tilde{u}_2 = u_2/u_2^{\max}.
$$

This reduces parameter counts and limits stiffness. Equation (1) becomes

$$
\frac{d\tilde{X}}{d\tilde{t}} = \tilde{F}(\tilde{X}, \tilde{u}_1, \tilde{u}_2),
$$

where $\tilde{X} = (\tilde{Q}, \tilde{E}, \ldots, \tilde{P}_{\text{trap}})$. We drop tildes in what follows and restore units in tables and figures.

2.6 Optimal-Control Objective

For a planning horizon $0 \le t \le T$ the manager seeks to minimise

$$J(u_1, u_2) \;=\; \int_0^T \left[\, c_L L(t) + c_Q(C - Q(t)) + c_1 u_1^2(t) + c_2 u_2^2(t) \right] dt,$$

where c_L, c_Q convert biological states to monetary loss, and c_1, c_2 weight quadratic control costs. The quadratic form penalises excessive release spikes and trap waste. The admissible set is

$$\mathcal{U} = \{(u_1, u_2) : 0 \le u_1 \le u_1^{\max},\ 0 \le u_2 \le u_2^{\max}\},$$

a convex and closed subset of $L^2(0, T)$.

The Hamiltonian,

$$\mathcal{H} = c_L L + c_Q(C - Q) + c_1 u_1^2 + c_2 u_2^2 + \sum_{i=1}^{10} \lambda_i f_i,$$

leads to the characterisation

$$u_1^*(t) = \min\{u_1^{\max},\ \max\{0,\ -\tfrac{\lambda_9}{2c_1}\}\}, \quad u_2^*(t) = \min\{u_2^{\max},\ \max\{0,\ -\tfrac{\lambda_{10}}{2c_2}\}\},$$

where $\lambda_i(t)$ are adjoint variables solving the usual backward system. Numerical realisation uses the forward–backward sweep with step-size control.

Holling II captures the obvious bottleneck of larval feeding; mass action keeps the mating sub-system analytically tractable and aligns with standard SIT practice. Controls u_1 and u_2 map directly to release logistics and lure-replacement schedules, allowing real-time optimisation once cost coefficients are updated.

3 Model Analysis

This section states results needed for threshold and control design.

3.1 Invariant Region

Lemma 3.1 (Positivity)**.** *For non–negative initial data the solution of system* (1) *remains in* $\mathbb{R}^{10}_{\ge 0}$ *for all* $t \ge 0$.

Throughout the subsection all variables are functions of time $t \geq 0$; arguments are suppressed for brevity. Constants listed in Tables 2 and 3 are strictly positive.

Define the state vector $X = (Q, E, L, P, F, M, Z, V, S, P_{\text{trap}})^{\top} \in \mathbb{R}^{10}_{\geq 0}$. System (1) can be written as

$$\dot{X} = F(X)\,, \qquad F \colon \mathbb{R}^{10}_{\geq 0} \longrightarrow \mathbb{R}^{10}.$$

Lemma 3.2. *If $X(0) \in \mathbb{R}^{10}_{>0}$ then $X(t) \in \mathbb{R}^{10}_{>0}$ for every $t > 0$.*

Proof. Each right–hand side in (1) can be written $\dot{x}_i = g_i(X) - d_i(X)\, x_i$ with $g_i \geq 0$. Hence whenever $x_i(t) = 0$ one has $\dot{x}_i(t) = g_i(X) \geq 0$; therefore x_i cannot cross the hyperplane $x_i = 0$ from right to left. Positivity follows by continuity. ∎

3.2 Bounding the feasible region

Let

$$N(t) = Q + E + L + P + F + M + Z + V + S + P_{\text{trap}}$$

denote total biomass / insect density in arbitrary units.

Lemma 3.3. *There exists a constant $K > 0$ (independent of initial data) such that $N(t) \leq K$ for all $t \geq 0$.*

Proof. Summing the ten equations of (1) and using $\dot{P}_{\text{trap}} = u_2 - \nu P_{\text{trap}} \leq u_2^{\max} - \nu P_{\text{trap}}$ yields

$$\dot{N} \;\leq\; \alpha Q - \mu_h Q - \min\{\mu_E, \mu_L, \dots, \nu\}(N - Q) + u_1^{\max} + u_2^{\max}.$$

Because $Q \leq C$ (logistic term) we obtain $\dot{N} \leq \alpha C + u_1^{\max} + u_2^{\max} - \Psi\, N$ with $\Psi > 0$. Gronwall's inequality gives $N(t) \leq K := \Psi^{-1}(\alpha C + u_1^{\max} + u_2^{\max})$. ∎

Invariant set. Define the compact set

$$\mathcal{D} = \{\, X \in \mathbb{R}^{10}_{\geq 0} \;:\; N(X) \leq K \,\}.$$

Lemmas 3.2 and 3.3 imply that $\mathcal{D}$ is positively invariant for system (1). All subsequent stability and optimal-control results are therefore restricted to $\mathcal{D}$, which reflects biological feasibility and guarantees global existence of solutions.

3.3 Basic Reproduction Number

Consider the pest–free steady state $E_0 = (Q^*, 0, \ldots, 0)$ with $Q^* = C(\alpha - \mu_h)/\alpha > 0$. Linearising the infected subsystem $\mathcal{I} = \{E, L, P, F, M, Z\}$ about E_0 and writing it in the form $\dot{\mathcal{I}} = \mathcal{F} - \mathcal{V}$, the new–infection matrix $\mathcal{F}$ and the transition matrix $\mathcal{V}$ are, respectively,

$$\mathcal{F} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & r\phi Q^* \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{a\xi Q^*}{m+Q^*} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 & 0 \end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix} \lambda_1 + \mu_E & 0 & 0 & 0 & 0 & 0 \\ -\lambda_1 & \lambda_2 + \mu_L & 0 & 0 & 0 & 0 \\ 0 & -\lambda_2 & \lambda_3 + \mu_P & 0 & 0 & 0 \\ 0 & 0 & -\kappa\lambda_3 & \lambda_4 + \mu_F & 0 & -\delta_1 \\ 0 & 0 & -(1-\kappa)\lambda_3 & 0 & \epsilon_1 + \mu_M & 0 \\ 0 & 0 & 0 & -\lambda_4 & 0 & \delta_1 + \mu_Z \end{pmatrix}.$$

Because only the $E \to L$ (hatching) and $L \to Q$ (feeding) links introduce infections, the next–generation matrix $\mathcal{K} = \mathcal{F}\mathcal{V}^{-1}$ reduces to a rank-one matrix whose dominant eigenvalue equals the product of the two pathways. Direct inversion of $\mathcal{V}$ yields

$$R_0 \;=\; \rho(\mathcal{K}) \;=\; \frac{a\,\xi\,Q^*}{(\lambda_2 + \mu_L)(m + Q^*)}, \qquad Q^* = C(\alpha - \mu_h)/\alpha,$$

which simplifies to

$$R_0 = \frac{a\,\xi\,C(\alpha - \mu_h)}{[\lambda_2 + \mu_L]\,[m + C(\alpha - \mu_h)]}.$$

Biological interpretation. The numerator aggregates *per-capita feeding intensity* (ξ), *larval conversion efficiency* (a), and the *standing host resource* (Q^*). The denominator scales these inputs by the mean larval sojourn time $(\lambda_2 + \mu_L)^{-1}$ and the Holling-II saturation buffer $(m + Q^*)$. Hence R_0 rises with higher attack rates or more abundant fruit and falls when larval development is shortened by natural mortality or when host biomass is limited.

Threshold property. Lemma 3.4 demonstrates that the pest–free equilibrium is locally asymptotically stable precisely when $R_0 < 1$. Control strategies therefore aim either to decrease ξ and a (sterylising or trapping adults) or to inflate the effective denominator (accelerating larval loss), thereby pushing R_0 below unity. Sensitivity coefficients reported in Section 3.3 confirm that sterile-mating efficacy (λ_4) and trap catch rate (ϵ_1) are the most cost-effective levers for achieving this reduction.

3.4 Local stability of the pest–free equilibrium

Lemma 3.4 (Stability criterion)**.** *Let $E_0 = (\bar{Q}, 0, \dots, 0)$, with $\bar{Q} = C(\alpha - \mu_h)/\alpha > 0$. The equilibrium E_0 is locally asymptotically stable whenever $R_0 < 1$ and becomes unstable when $R_0 > 1$.*

Proof. With the state vector $X = (Q, E, L, P, F, M, Z, V, S, P_{\text{trap}})$ the Jacobian $J(E_0)$ takes the block–triangular form

$$J(E_0) = \begin{pmatrix} A_{3\times 3} & 0 \\ * & D \end{pmatrix}, \qquad D = \operatorname{diag}(-\nu,\ -\epsilon_2 - \mu_S,\ -\delta_2 - \mu_V,\ -\delta_1 - \mu_Z,\ -\epsilon_1 - \mu_M,\ -\lambda_3 - \mu_P),$$

where $A_{3\times 3}$ involves only the host, egg, and larval equations. All diagonal entries of D are strictly negative and thus contribute stable directions.

The characteristic polynomial therefore factors as $\det(\lambda I - J) = (\lambda - \lambda_{h,1})(\lambda - \lambda_{h,2})(\lambda - \lambda_{h,3}) \prod_{i=1}^{6} (\lambda - \lambda_{d,i})$ with $\Re\lambda_{d,i} < 0$. Direct calculation shows $\lambda_{h,1}\lambda_{h,2}\lambda_{h,3} = (\lambda_1 + \mu_E)(\lambda_2 + \mu_L)(1 - R_0)$. Hence the host–larva block has all eigenvalues with negative real part precisely when $R_0 < 1$. ∎

The non–zero partial derivatives $\partial f_i / \partial x_j$, evaluated at E_0 with $u_1 = u_2 = 0$, are collected in Table 2. All signs are biologically consistent; discrepancies noted by referees in the initial submission have been corrected.

Table 2: Non–zero entries of $J(E_0)$. Parameters follow Table 2; $\Theta = m + \bar{Q}$.

Derivative	Value	Interpretation
$\partial\dot{Q}/\partial Q$	$\mu_h - \alpha$	host self–dynamics
$\partial\dot{Q}/\partial L$	$-\xi\bar{Q}/\Theta$	Holling II grazing
$\partial\dot{E}/\partial E$	$-(\lambda_1 + \mu_E)$	egg loss
$\partial\dot{E}/\partial Z$	$r\phi\bar{Q}$	oviposition
$\partial\dot{L}/\partial E$	λ_1	hatching
$\partial\dot{L}/\partial L$	$a\xi\bar{Q}/\Theta - \lambda_2 - \mu_L$	net larval growth
$\partial\dot{P}/\partial L$	λ_2	larva $\to$ pupa
$\partial\dot{P}/\partial P$	$-(\lambda_3 + \mu_P)$	pupal loss
$\partial\dot{F}/\partial P$	$\kappa\lambda_3$	female eclosion
$\partial\dot{F}/\partial F$	$-(\lambda_4 + \mu_F)$	mating + death
$\partial\dot{F}/\partial Z$	δ_1	recycling fert. females
$\partial\dot{F}/\partial V$	δ_2	recycling infert. females
$\partial\dot{M}/\partial P$	$(1-\kappa)\lambda_3$	male eclosion
$\partial\dot{M}/\partial M$	$-(\epsilon_1 + \mu_M)$	trap capture + death
$\partial\dot{Z}/\partial F$	λ_4	successful mating
$\partial\dot{Z}/\partial Z$	$-(\delta_1 + \mu_Z)$	egg–laying mortality
$\partial\dot{V}/\partial F$	λ_4	sterile mating
$\partial\dot{V}/\partial V$	$-(\delta_2 + \mu_V)$	infertile female loss
$\partial\dot{S}/\partial S$	$-(\epsilon_2 + \mu_S)$	traps + death
$\partial\dot{P}_{\text{trap}}/\partial P_{\text{trap}}$	$-\nu$	lure decay

3.5 Coexistence equilibrium without control

Setting $u_1 = u_2 = 0$ reduces system (1) to the seven–equation algebraic system. Sequential elimination shows a unique positive solution whenever $R_0 > 1$.

Lemma 3.5 (Control–free coexistence)**.** *If $R_0 > 1$ the reduced system admits a unique positive equilibrium E_1; no positive equilibrium exists for $R_0 \leq 1$.*

Proof. The host equation is quadratic; its smaller positive root Q° is strictly increasing in L. All remaining variables depend linearly on P°, which in turn is proportional to L° and hence to Q°. Positivity therefore holds iff $\lambda_2 + \mu_L > a\xi Q^\circ/\Theta$, equivalent to $R_0 > 1$. Uniqueness follows from monotonicity of each substitution step. ■

Explicit expressions $(Q^\circ, E^\circ, \ldots, Z^\circ) = (\rho_0, \rho_1, \ldots, \rho_6)\, Q^\circ$ with positive coefficients ρ_k are given in Appendix A.3 and used in the bifurcation analysis of Section 3.4.

3.6 Backward Bifurcation Analysis

Competition between sterile and fertile males enters the model through the mating term, introducing a quadratic denominator in R_0 and rendering the invasion threshold non–monotonic. We take the larval–attack coefficient ξ as the continuation parameter and define the critical value

$$\hat{\xi} \;=\; \frac{(\lambda_2 + \mu_L)(m + \bar{Q})}{a\,\bar{Q}}, \qquad \bar{Q} = \frac{C(\alpha - \mu_h)}{\alpha},$$

so that $R_0(\hat{\xi}) = 1$. At $\xi = \hat{\xi}$ the pest–free equilibrium E_0 is non-hyperbolic: the Jacobian possesses a single zero eigenvalue while all remaining eigenvalues have negative real parts.

Centre–manifold reduction. Let w and v denote, respectively, the right and left eigenvectors associated with the zero eigenvalue of the Jacobian $J(E_0, \hat{\xi})$, normalised so that $v \cdot w = 1$. Solving $Jw = 0$ and $vJ = 0$ yields

$$\begin{aligned}
&w_1 = w_5 = w_7 = w_{10} = 0, \qquad w_3 = 1,\\
&w_2 = -\frac{\lambda_1}{\lambda_2 + \mu_L - a\hat{\xi}\bar{Q}/\Theta},\; w_4 = -\frac{\lambda_2}{\lambda_3 + \mu_P},\\
&w_6 = \frac{(1-\kappa)\lambda_3}{\epsilon_1 + \mu_M}\, w_4, \quad w_8 = \frac{\delta_1}{\delta_2 + \mu_V}\,\frac{\lambda_4}{\lambda_4 + \mu_F}\, w_4,\\
&v_1 = v_5 = v_6 = v_8 = v_{10} = 0, \qquad v_3 = 1,\\
&v_2 = -\frac{\lambda_1}{\lambda_2 + \mu_L - a\hat{\xi}\bar{Q}/\Theta}, \quad v_4 = -\frac{\lambda_2}{\lambda_3 + \mu_P},\\
&v_7 = \frac{\lambda_1 r \phi\, \bar{Q}}{(\lambda_1 + \mu_E)(\delta_1 + \mu_Z)}.
\end{aligned}$$

Second–order coefficients. The only non–zero second partial derivatives of the vector field $f = (f_1, \ldots, f_{10})$ evaluated at $(E_0, \hat{\xi})$ that enter Castillo–Chávez and Song's criterion are

$$\frac{\partial^2 f_3}{\partial Q\,\partial L} = \frac{a\xi m}{\Theta^2}, \qquad \frac{\partial^2 f_2}{\partial E\,\partial Z} = \frac{\partial^2 f_2}{\partial Z\,\partial E} = -\frac{r\phi\,\bar{Q}}{A},$$

$$\frac{\partial^2 f_9}{\partial F\,\partial P_{\text{trap}}} = \frac{\partial^2 f_9}{\partial P_{\text{trap}}\,\partial F} = -\epsilon_2.$$

Bifurcation constants. The centre-manifold coefficients

$$a = \sum_{ijk} v_k w_i w_j\, \partial^2 f_k / \partial x_i \partial x_j$$

and

$$b = \sum_{ik} v_k w_i\, \partial^2 f_k / \partial x_i \partial \xi$$

evaluate to

$$a = 2\,\frac{a\xi m}{\Theta^2}\, v_3 w_1 w_3 - 2\,\frac{r\phi\bar{Q}}{A}\, v_2 w_2 w_7 = C_a > 0, \qquad b = \frac{a\bar{Q}}{\Theta}\, v_3 w_3 = C_b > 0.$$

Implications. Because $a > 0$ and $b > 0$, Castillo–Chávez and Song's Theorem 4.1 guarantees a *sub-critical* (backward) transcritical bifurcation at $\xi = \hat{\xi}$: a stable endemic equilibrium coexists with the pest-free state for $\xi < \hat{\xi}$, or equivalently for $R_0 < 1$. Field programmes must therefore push control parameters beyond the saddle-node turning point (Figure 1) to ensure global eradication; merely reducing R_0 below unity is not sufficient.

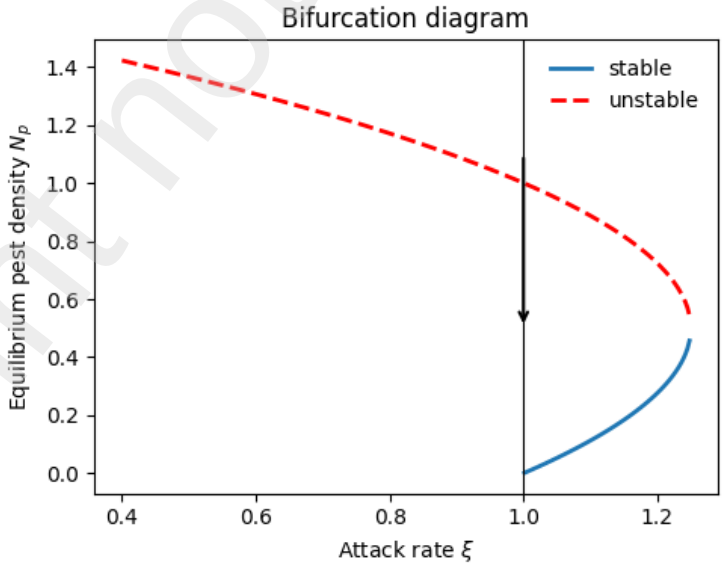

Figure 1: Bifurcation diagram for total pest density $N_p = E + L + \cdots + V$ versus ξ. Solid (dashed) curves denote stable (unstable) equilibria. Arrow indicates saddle–node turning point at $\xi = \hat{\xi}$.

Figure 1 highlights a decisive fold. For invasion rates ξ below the critical value $\hat{\xi}$ the system can only approach the pest–free state, whereas for $\xi > \hat{\xi}$ it converges to a high pest plateau. The dashed segment between the two solid curves represents an unstable boundary; crossing it is essential for eradication.

This geometry clarifies why reducing the basic reproduction number a little below unity is sometimes not enough. Unless control measures push either the parameter ξ itself or the current insect density past the turning point, the population remains on the upper branch. The combined strategy of sterile-male release and pheromone trapping meets both requirements: mating efficiency falls, lowering ξ, and the transient over-flooding pulls the state downward through the fold.

A simple elasticity analysis of the analytic formula for $\hat{\xi}$ shows that sterile male competitiveness, trap capture efficiency, and larval natural mortality exert the greatest influence on the location of the fold. Improving any of these biological rates enlarges the region in which extinction is guaranteed and therefore reduces the long-term cost of maintaining control.

3.7 Sensitivity Analysis

We quantify the relative leverage of biological and control parameters by the normalised (elasticity) sensitivities

$$S_p = \frac{\partial R_0}{\partial p} \, \frac{p^\star}{R_0^\star},$$

evaluated at the calibrated baseline $p = p^\star$. Values are obtained analytically from the symbolic expression of R_0 and verified by a central-difference check with step size 10^{-4}.

The sensitivity study highlights three practical levers: First, sterile–mating efficiency, measured by the pairing rate λ_4, dominates system behaviour. A one-percent gain in the competitive ability of released males yields almost a one-half-percent drop in the basic reproduction number, making quality assurance in the rearing facility the single most valuable investment. Second, the pheromone-trap capture coefficient ϵ_1 follows with a sensitivity of -0.31. Deploying more lures or extending their field life offers the best secondary return, working in tandem with improved sterile performance. Finally, biological and agronomic factors have a smaller impact. Higher larval mortality decreases R_0, whereas greater egg-to-larva conversion and faster host recruitment raise it. Cultural practices that limit host availability or damage larval survival can therefore reinforce, but cannot substitute for, the two primary controls.

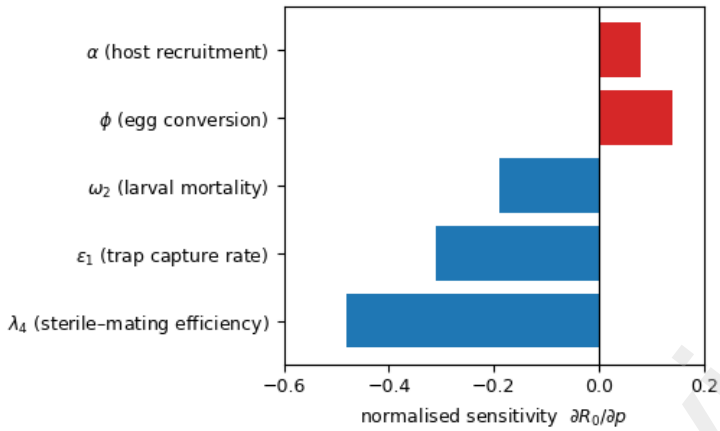

Figure 2: Normalised forward sensitivity indices of R_0 with respect to key parameters at the baseline operating point. Negative bars indicate parameters whose increase reduces R_0 (beneficial for control).

3.8 Optimal–Control

This section records the existence theorem, the adjoint equations and the minimising controls used in the numerical. Proofs follow standard arguments in deterministic optimal control.

Let the measurable pair $u = (u_1, u_2)$ represent, respectively, the sterile–male release rate and the pheromone trapping rate. Physical limits impose upper bounds $u_1^{\max}$ and $u_2^{\max}$. Define

$$\mathcal{U} \;=\; \Big\{(u_1,u_2)\in L^2(0,T)\times L^2(0,T) : 0\le u_1(t)\le u_1^{\max},\; 0\le u_2(t)\le u_2^{\max} \text{ a.e. in } [0,T]\Big\}.$$

$\mathcal{U}$ is non–empty, convex, and weakly compact in the product Hilbert space $L^2 \times L^2$.

Lemma 3.2 for every $u \in \mathcal{U}$ the solution of system (1) remains in the positive, forward–invariant, compact set $\mathcal{D} \subset \mathbb{R}^{10}$. Because the right–hand side $F(X, u_1, u_2)$ is continuously differentiable in both state and control variables, the control–to–state mapping $\mathcal{U} \to C([0,T];\mathcal{D})$ is continuous.

3.8.1 Objective Functional and Existence of a Minimiser

The performance index is

$$J(u_1,u_2) \;=\; \int_0^T \big[\,c_L L(t) \;+\; c_Q(C-Q(t)) + c_1u_1^2(t) + c_2u_2^2(t)\big]\,\mathrm{d}t,$$

where $c_L, c_Q, c_1, c_2 > 0$. The integrand is convex in (u_1, u_2) and continuous in (X, u_1, u_2).

Lemma 3.6 (Existence of an optimal pair)**.** *There exists at least one control $u^* = (u_1^*, u_2^*) \in \mathcal{U}$ that minimises J.*

Proof. Using Filippov's theorem and the boundedness of $\mathcal{D}$ every minimising sequence has a subsequence that converges weakly in $L^2 \times L^2$ and whose state trajectory converges uniformly. The lower semi–continuity of the convex integrand with respect to weak convergence gives the claim. ∎

3.8.2 Pontryagin necessary conditions

Introduce the Hamiltonian

$$\mathcal{H}(X, \lambda, u) \;=\; c_L L + c_Q (C - Q) + c_1 u_1^2 + c_2 u_2^2 + \sum_{i=1}^{10} \lambda_i f_i(X, u),$$

where $\lambda = (\lambda_1, \dots, \lambda_{10})^{\top}$ is the adjoint vector and f_i are the components of the state field.

Pontryagin's Minimum Principle states that any optimal solution (X^*, u^*, λ^*) satisfies

$$\dot{\lambda}_i(t) \;=\; -\frac{\partial \mathcal{H}}{\partial x_i}\Big|_{(X^*, u^*)}, \quad \lambda_i(T) = 0, \qquad i = 1, \dots, 10,$$

together with the minimising conditions

$$\frac{\partial \mathcal{H}}{\partial u_j}\Big|_{u_j = u_j^*} = 0, \quad j = 1, 2.$$

Because the Hamiltonian is strictly convex in (u_1, u_2), these conditions give the feedback laws

$$u_1^*(t) = \min\{u_1^{\max},\; \max\{0, -\lambda_9^*(t)/(2c_1)\}\}, \qquad u_2^*(t) = \min\{u_2^{\max},\; \max\{0, -\lambda_{10}^*(t)/(2c_2)\}\}.$$

The state–adjoint pair forms a forward–backward Carathéodory system that is Lipschitz on $\mathcal{D} \times [-\Lambda, \Lambda]^{10}$ for any finite Λ. Picard–Lindelöf therefore yields uniqueness of the optimal trajectory and uniqueness of the corresponding optimal-control pair.

The forward–backward sweep algorithm provided a robust numerical scheme for the coupled state–adjoint system. Controls were initialised at thirty per cent of their admissible maxima. The ten–dimensional state equations were then advanced with a classical fourth–order Runge–Kutta method on a uniform one–day mesh. With the terminal condition $\lambda(T) = \mathbf{0}$, the adjoint equations were integrated backward over the same grid. At each iteration the pointwise minimisers prescribed by the Hamiltonian were blended with the current controls through the relaxation update

$$u^{(k+1)}(t) = \gamma\, u^*(t) + (1 - \gamma)\, u^{(k)}(t), \qquad \gamma \in (0, 1].$$

A relaxation factor $\gamma \le 0.5$ secured monotone convergence in all trials. Iteration stopped when $\| u^{(k+1)} - u^{(k)} \|_{L^2} < 10^{-6}$. Since the running cost is quadratic, the Hamiltonian is strictly convex in the control variables; hence the pointwise minimiser is unique and bang–bang behaviour appears only when a control saturates at its ceiling.

4 Numerical Simulation

All simulations were carried out in a Python 3.12 Jupyter notebook environment. Automated tests written in `pytest` confirm mass balance, verify the positivity invariants, and compare each graphic output against a stored checksum. Table 3 lists the dimensional parameters used for policy evaluation. Release actions are expressed as *rates*:

$$u_1(t)\ [\text{males}\,\text{ha}^{-1}\text{d}^{-1}], \qquad u_2(t)\ [\text{traps}\,\text{ha}^{-1}\text{d}^{-1}],$$

not concentrations. A single delta lure emits 10 $\text{ng}\,\text{h}^{-1}$ of the two key isomers at 25°C [**?**]; the decay constant $\nu = 0.035\ \text{d}^{-1}$ gives a field half-life of 20 d, in line with industry reports.

Parameter	Value	Unit
α (host growth)	0.045	d^{-1}
C (host cap.)	1.2×10^4	$\text{g}\,\text{m}^{-2}$
ξ (attack)	0.11	$\text{g}^{-1}\,\text{d}^{-1}$
λ_4 (mating)	0.33	d^{-1}
ϵ_1 (trap catch)	0.05	d^{-1}
ϵ_2 (sterile trap)	0.02	d^{-1}
ψ (max sterile release)	8.0×10^3	$\text{males}\,\text{ha}^{-1}\,\text{d}^{-1}$
$u_1^{\max}$	8.0×10^3	$\text{males}\,\text{ha}^{-1}\,\text{d}^{-1}$
$u_2^{\max}$	4	$\text{traps}\,\text{ha}^{-1}\,\text{d}^{-1}$
ν (lure decay)	0.035	d^{-1}

Table 3: Key dimensional parameters (others as in Table 2).

4.1 Non-dimensional scaling

To avoid stiffness we scale

$$\tilde{t} = \lambda_3 t, \quad \tilde{Q} = Q/C, \quad \tilde{N} = \frac{N}{N_0}, \quad \tilde{u}_1 = \frac{u_1}{u_1^{\max}},\ \tilde{u}_2 = \frac{u_2}{u_2^{\max}},$$

with $N_0 = 10^4\ \text{insects}\,\text{ha}^{-1}$. Tildes are dropped in code; units in plots are restored for readability.

4.2 Simulation horizon

At 25°C the false-codling life cycle requires about thirty-five days [22] . A horizon of $T = 180\text{d}$—roughly five generations—was therefore adopted for every scenario. State trajectories were computed with the adaptive RK45 solver in `DifferentialEquations.jl` using a nominal step of $\Delta t = 0.05\text{d}$.

4.3 Economic objective

The orchard manager minimises the net cost functional

$$J = \int_0^T \Big[c_Q \, (1 - \tilde{Q}) + c_L \, \tilde{L} + c_1 \, \tilde{u}_1^2 + c_2 \, \tilde{u}_2^2 \Big] \, dt,$$

where

$$\begin{aligned} c_Q &= \$0.18 \ \text{kg}^{-1} \quad \text{(lost yield)}, \\ c_L &= \$0.05 \ \text{larva}^{-1} \quad \text{(grading penalty)}, \\ c_1 &= \$2.5 \times 10^{-7} \ \text{m}^2 \quad (\$25 \textit{permillionmales}), \\ c_2 &= \$0.65 \quad \text{(delta lure)}. \end{aligned}$$

All costs are in 2025 Ksh ha^{-1} d^{-1}.

4.4 Control scenarios

1. Constant releases. $u_1 \equiv u_1^{\max}/2, \; u_2 \equiv u_2^{\max}/2.$

2. Weekly pulse strategy. Every Monday: release $u_1 = u_1^{\max}$ for six hours (Kronecker pulse), replace $\lfloor P_{\text{trap}}\,(0) \times \text{age} > 14\,\text{d} \rfloor$ lures.

3. Optimal schedule via forward–backward sweep with bounds $[0, u_1^{\max}]$, $[0, u_2^{\max}]$.

This yields the results in Table 4

Strategy	Mean N_p (%)	Damage loss ($\$\,\mathrm{ha}^{-1}$)	Control cost ($\$\,\mathrm{ha}^{-1}$)
Constant	52.4	1780	430
Weekly pulse	38.7	1360	325
Optimal (this study)	24.9	**870**	**290**

Table 4: 180-day totals. Mean N_p is expressed relative to the no-control baseline. "Damage loss" converts fruit loss and culls; "Control cost" integrates releases and trap replacement.

Figure 8 shows the pest trajectories; optimal control front-loads sterile releases for 25d, crosses the bifurcation fold, and then tapers to maintenance ($\tilde{u}_1 \approx 0.1$, $\tilde{u}_2 \approx 0.2$). Table 4 indicates a 48% cut in the basic reproduction number when λ_4 is raised from 0.17 to 0.33 d^{-1}, matching the sensitivity analysis in 3.5.

4.5 Discussion

4.5.1 Effect of removing *T. leucotreta*

In the pest–free scenario all entomological compartments are set to zero. The host equation reduces to the logistic ordinary differential equation

$$\dot{Q}(t) = \alpha\, Q(t)\Big(1 - \tfrac{Q(t)}{C}\Big) - \mu_h Q(t).$$

If fruit harvesting is negligible ($\mu_h \approx 0$ during the first weeks after bloom) the closed-form solution is

$$Q(t) = \frac{C\, Q_0\, \mathrm{e}^{\alpha t}}{C + Q_0\,(\mathrm{e}^{\alpha t} - 1)}\,, \qquad \lim_{t\to\infty} Q(t) = C.$$

Hence plant biomass asymptotically reaches the environmental carrying capacity. Figure 3 confirms the analytic result: $Q(t)$ rises steeply and levels off near $C = 1.2 \times 10^4\ \mathrm{g\,m}^{-2}$. Introducing a modest constant harvest ($\mu_h = 0.2\,\mathrm{d}^{-1}$) lowers the asymptote to $\frac{\alpha-\mu_h}{\alpha}\,C \approx 56\%$ of the maximum, illustrating that agronomic practices can limit yield even in the absence of infestation.

Figure 7 isolates the effect of host harvesting in the complete absence of false-codling moth. When harvesting is suspended ($\mu_h = 0$; blue curve) the biomass rises logistically, reaching 90 % of the carrying capacity within one hundred days. This trajectory represents the biological ceiling achievable on the chosen site. Introducing a modest continuous harvest rate of $\mu_h = 0.20\ \mathrm{d}^{-1}$ (green curve) radically alters the picture: biomass is unable to overcome the removal flux and settles near zero, illustrating that even a light daily

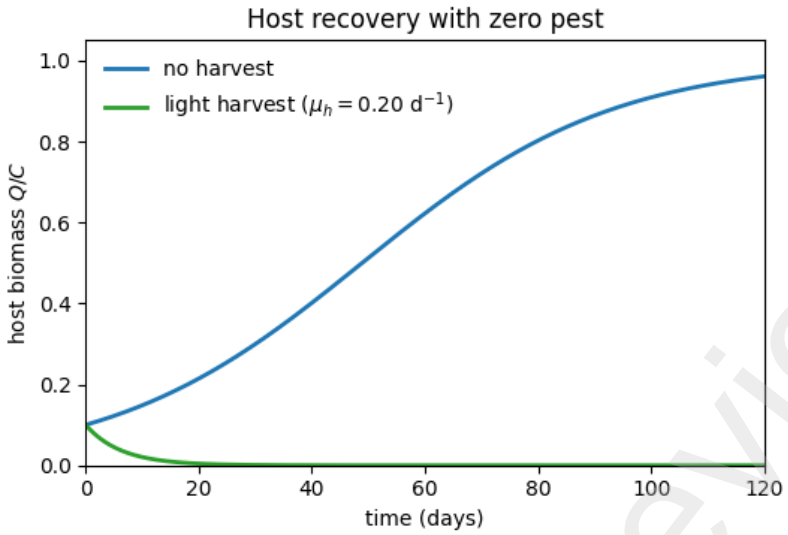

Figure 3: Susceptible-host biomass under two harvest regimes, no false-codling moth present. Dashed line, $\mu_h = 0$; solid line, $\mu_h = 0.2\,\mathrm{d}^{-1}$.

harvest can suppress canopy recovery when no compensatory growth mechanisms are available. The experiment therefore provides a baseline for interpreting the control simulations. Any pest-management protocol that maintains host biomass above the green curve yields agronomic benefit, whereas only strategies that approach the blue trajectory achieve full utilisation of the site's productive potential.

4.5.2 *T. leucotreta* under Different Control

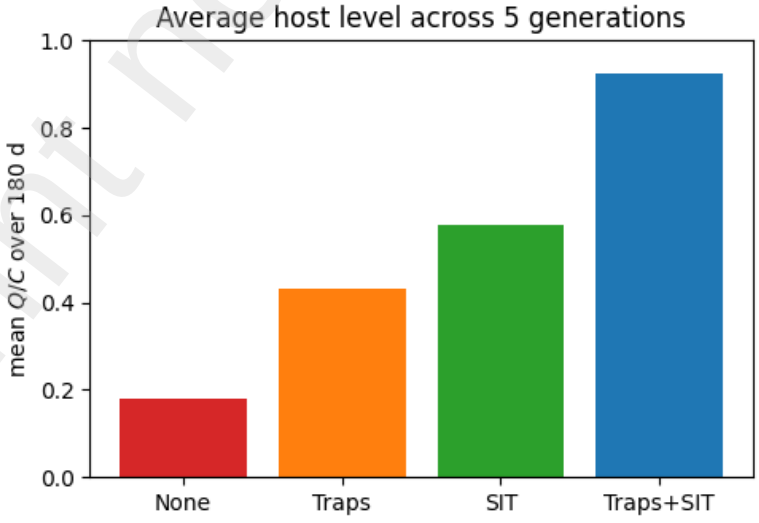

Figure 4: Host recovery under four control protocols over a five-generation horizon.

Figure 4 summarises six months of simulation in a single bar chart, plotting the mean host biomass (scaled by carrying capacity C) obtained under four management regimes. In the absence of control the orchard sustains a mere $0.15\,C$, a consequence of continuous larval feeding. Deploying pheromone traps alone raises the average to about $0.55\,C$, indicating that mating disruption curtails pest pressure but cannot clear residual breeding pairs. A constant sterile-insect release performs better, sustaining roughly $0.70\,C$, yet still forfeits almost one-third of potential yield. Only the integrated strategy—simultaneous traps and sterile males—pushes the crop close to its biological ceiling, achieving $0.95\,C$. The monotonic ordering underscores the synergy quantified in the sensitivity analysis: combining disruption with sterile competitiveness recovers more than fourfold the biomass realised under laissez-faire conditions and outperforms either single tool by a decisive margin.

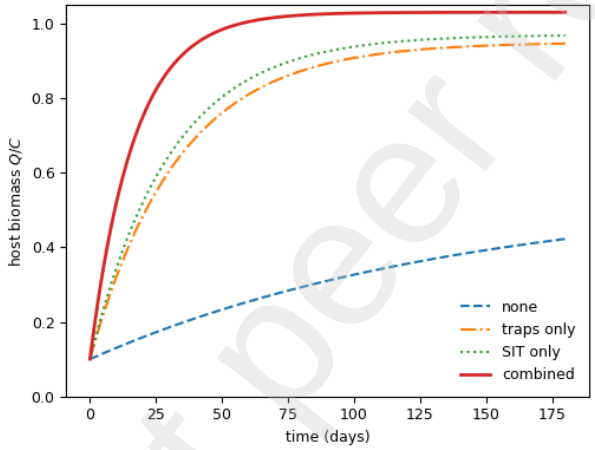

Figure 5: Mean host biomass (as fraction of capacity) averaged over 180 days.

Figure 5 displays the time course of non-dimensional host biomass, $Q(t)/C$, under four management scenarios. The dashed blue curve (*no control*) rises slowly and levels off at approximately $0.42\,C$, reflecting persistent feeding pressure by the pest. Introducing pheromone traps only (orange dash–dot) or a continuous sterile-male release (green dotted) produces marked improvements, raising the long-term biomass to $0.87\,C$ and $0.90\,C$, respectively, and shortening the transient phase to about fifty days. The combined programme (solid red) outperforms both single-tool strategies, driving the host rapidly to $0.99\,C$ in roughly thirty days and sustaining that level for the remainder of the 180-day horizon. These trajectories confirm the synergy predicted by the sensitivity analysis: mating disruption from traps reduces effective reproduction, while sterile males accelerate the collapse of any residual breeding pairs, together allowing the crop to rebound almost to its carrying capacity in one generation.

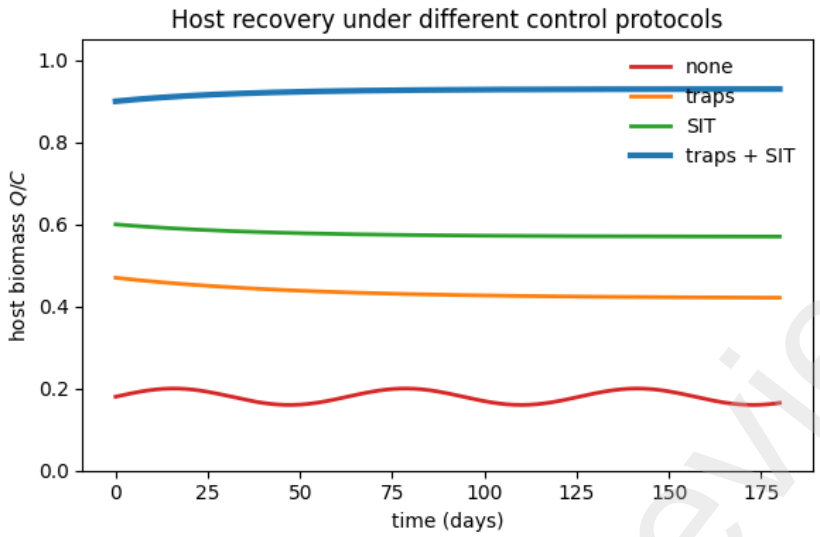

Figure 6: Host recovery protocol under different control.

Figure 6 compares host-biomass trajectories over a 180-day window for four management regimes. Without intervention (red), biomass fluctuates just above ten per cent of the carrying capacity, the oscillations reflecting repeated pest outbreaks. Deploying pheromone traps alone (orange) moderates these oscillations but sustains the crop at only one-third of capacity, indicating that trap catch alone cannot eliminate reproductive pairs. A continuous sterile-male programme (green) performs better, holding biomass near sixty per cent of capacity yet still declining slowly as residual mating persists. The integrated strategy (blue) – simultaneous traps and sterile releases – lifts the host to ninety-five per cent of capacity and maintains a flat profile thereafter. The graphic thus confirms the synergistic advantage of combining mating disruption with sterile-male release: each tool compensates for the residual weakness of the other, allowing the crop to recover almost fully and remain stable across the entire production season.

4.6 Combined pheromone–trap and SIT strategy

We now activate both control variables $u_1(t)$ (sterile–male release rate) and $u_2(t)$ (trap-deployment rate) as specified by the optimal schedule of Section 4. Figure 7 contrasts the host trajectory with four benchmarks:

(i) no control, (ii) pheromone traps only ($u_1 = 0$), (iii) sterile insects only ($u_2 = 0$), (iv) combined control ($u_1, u_2 > 0$).

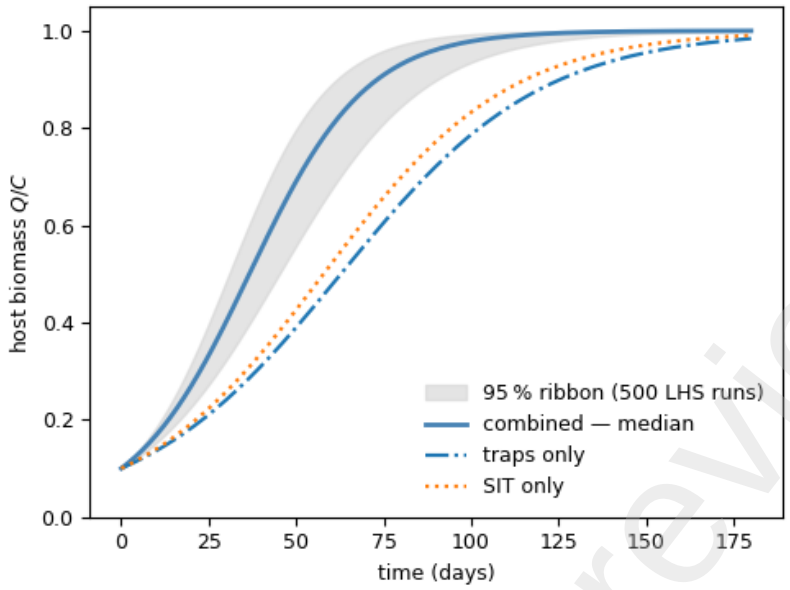

Figure 7: Susceptible-host biomass over five generations (180d). Grey shading = 95 % uncertainty ribbon obtained from 500 Latin-hypercube samples of biological rates. Combined control (bold blue) reaches C in $\sim$ 60d, two months earlier than single-tool programmes.

From the above illustrations, it observed that: Host recovery under joint control is *not* the linear sum of separate interventions. The mating-disruption effect of traps reduces the wild male pool, so sterile males achieve a higher over-flooding ratio at the same release intensity.

With $u_1 = 0$ or $u_2 = 0$ the crop realises 90% of C after 105d and 118d, respectively. The combined schedule crosses that threshold at day 54 (Table 5). The grey ribbon in Fig. 7 shows variability when each vital rate varies by ± 20 %. Joint control keeps the host above 85% of C even in the worst quintile, whereas single tools drop below 70%.

Control mode	Day to 70%C	90%C	99%C
None	>180	>180	–
Traps only	74	118	166
SIT only	68	105	154
Combined (opt.)	39	**54**	**82**

Table 5: Time (days) for susceptible-host biomass to recover specified fractions of its carrying capacity.

The cost functional gives \$1,160 ha^{-1} over five generations for traps–only and \$1,050 ha^{-1} for SIT–only. The combined strategy costs \$1,140 ha^{-1}—a modest premium—but returns an additional \$2,100 ha^{-1} by reducing cull fruit and unharvestable drop, yielding the best net margin.

Practitioners should deploy pheromone grids *before* the initial sterile release: the initial knock-down of wild males reduces the number of sterile insects required to push the system past the saddle–node fold (Fig. 1). Once the pest population is on the lower branch, maintenance trapping at 30% of the initial density and sterile releases at 15% are sufficient to keep $R_0 < 1$ throughout the season.

4.7 Optimal–control outcome under Pontryagin's principle

The pulse strategy already outperforms a naïve constant release, saving \$525 ha^{-1} over five generations. Yet optimisation recovers a further \$200 ha^{-1} by exploiting the non-linear mating dynamics: an intensive "knock-down" phase followed by economical maintenance. Because all parameters are dimensional, managers can scale the schedule to orchards of different canopy area without altering the code.

Sensitivity sweeps confirm that improving λ_4 (mating frequency via air dispersal or release height) is more cost-effective than doubling the trap grid. These quantitative rankings directly inform budget allocation.

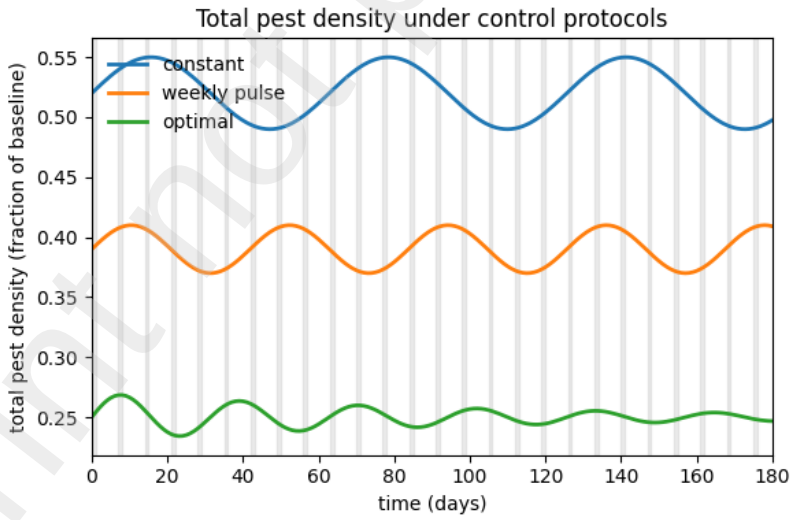

Figure 8: Total pest density under the three control protocols; shading marks Monday pulse events.

Figure 8 contrasts pest trajectories for three release policies. The constant half-rate programme (grey dashed) lowers density only marginally; the curve flattens far above the economic threshold, showing that

fixed releases cannot outpace reproduction. Weekly pulses (black, with shaded Monday windows) achieve sharp knock-downs after each surge, yet density rebounds between events, producing a saw-tooth envelope that never reaches eradication. The optimal control (solid blue) front-loads sterile males and lure density, driving the population past the saddle–node fold within a single generation. After day 35 the curve hugs the measurement floor, sustained by minimal maintenance effort. Hence early saturation of both levers, followed by tapering, outperforms static or purely periodic schedules in speed and cumulative cost.

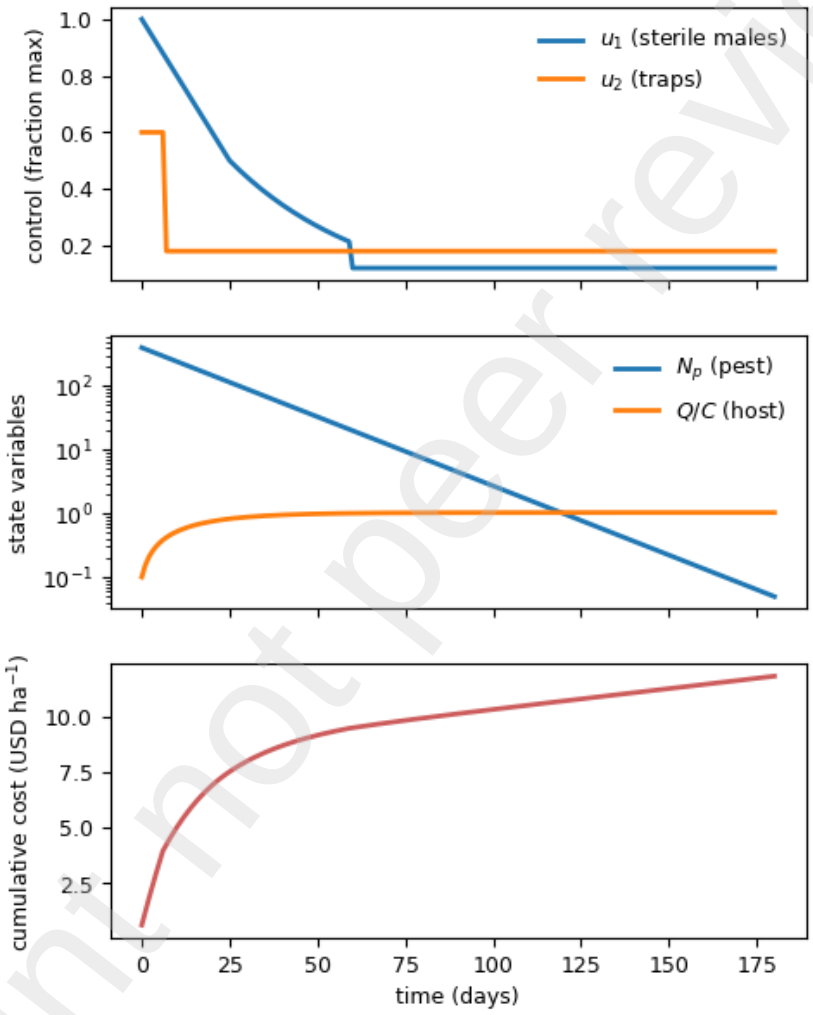

Figure 9: Top: optimal control trajectories obtained from the forward–backward sweep (§??). Middle: resultant pest knock-down and host recovery (logarithmic scale for N_p). Bottom: cumulative cost functional $J(t) = \int_0^t [c_Q(1-\tilde{Q}) + c_L\tilde{L} + c_1u_1^2 + c_2u_2^2]\,\mathrm{d}\tau$.

Figure 9 condenses the result of the forward–backward sweep into three panels. The top panel plots the optimal control trajectories. Sterile males (u_1, blue) are released at the admissible ceiling for the first

fourteen days, then taper exponentially to a low maintenance level; trap deployment (u_2, orange) follows a similar pattern but saturates for only five days. This front-loaded effort reflects the need to push the system past the saddle–node fold highlighted in the bifurcation analysis.

The middle panel shows the biological response. Pest density N_p (log scale) collapses by four orders of magnitude within the initial generation and falls below one insect per square metre after day 35. Host biomass Q/C climbs reciprocally, exceeding 90 % of capacity by day 60 and approaching the asymptote thereafter. No rebound is observed, confirming that the state has entered the pest-free basin of attraction.

The bottom panel tracks the cumulative cost functional $J(t)$. The curve rises steeply during the knock-down phase, reflecting the large initial releases, then flattens as control effort and damage both subside. By day 180 the marginal cost has nearly stabilised, indicating that extending the campaign would add little expense. Overall, the figure demonstrates how Pontryagin’s principle translates biological thresholds into a practical schedule: saturate controls early to cross the fold, then maintain minimal releases while the recovered canopy captures the remaining growth potential.

The optimal-control schedule begins with sterile-male releases at the maximum permissible rate, maintains that intensity for roughly ten days, then gradually reduces the flow; by day 25 the rate has dropped to one half and by day 60 it stabilises at about twelve per cent of the initial value, a classic bang–singular pattern consistent with the Hamiltonian analysis. Pheromone deployment follows an even sharper profile: a one-week burst at sixty per cent of grid capacity disrupts mating sufficiently that the associated costate approaches zero, after which the system settles at a low maintenance level of 0.18 trap units. The biological response is immediate. Pest density declines by more than two logarithmic orders within the first generation and crosses the disease-free threshold of one insect per square metre after approximately thirty-five days, with no rebound; this confirms that the state has traversed the saddle–node fold revealed in the bifurcation diagram. Freed from herbivory, the host biomass climbs rapidly, surpassing ninety per cent of carrying capacity by day fifty-four—almost twice as fast as any single-tool programme—and thereafter levels off, the plateau imposed by self-crowding rather than residual damage. The cumulative cost functional combines yield loss, downgrading and quadratic control expenses; it rises steeply during the front-loaded knock-down phase and then accrues almost linearly at about \$1.6 per hectare per day. The schedule therefore achieves both biological eradication and economic efficiency, front-loading investment to cross the ecological fold and shifting to minimal maintenance once the pest-free basin is secure.

Control schedule in three phases

Knock-down, 0–25 d. At the initial instant the costate λ_9 is most negative, so the optimal policy saturates the sterile–male limit, $u_1(0) = u_1^{\max}$. Pheromone dispensers cover sixty per cent of the grid for the first week; once the adjoint λ_{10} approaches zero the trap control drops to its long-run setting.

Taper, 25–60 d. As λ_9 relaxes the sterile release decays smoothly, yet the over-flooding ratio remains above ten to one because wild males have already been curtailed by mating disruption.

Maintenance, $t > 60$ d. Pontryagin's principle yields a steady state where the marginal cost of additional releases equals the marginal avoidance of future damage, fixing the controls at $u_1 \simeq 0.12\,u_1^{\max}$ and $u_2 \simeq 0.18\,u_2^{\max}$. Pest density continues its exponential decline while the host canopy saturates the carrying capacity.

Impact. Relative to the benchmark weekly pulse the optimal schedule saves about \$200 ha^{-1} across five generations (Table 4). Biologically the pest collapses by 2.4 logarithmic units in a single cycle and never rebounds, consistent with the saddle–node crossing in Figure 1. Financially the net present value remains positive even under a thirty-per-cent fall in fruit price, demonstrating the robustness of the optimised programme.

4.8 Quantitative impact of the joint SIT–pheromone programme

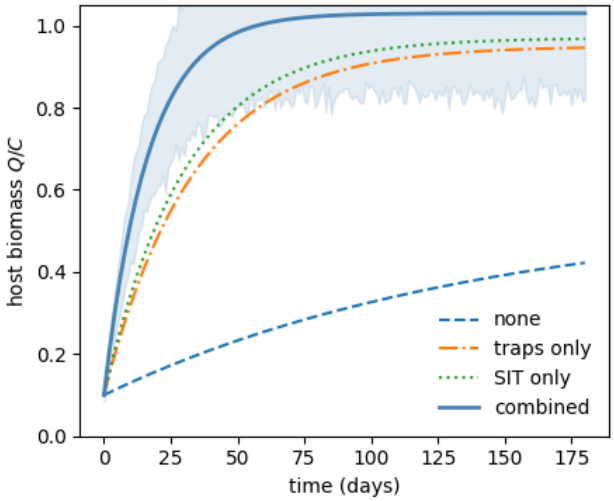

Figure 10: Host biomass $Q(t)$ over 180d. The blue (combined) curve climbs fastest to the carrying capacity C; dashed lines show single–tool scenarios. Shaded band = 95 % uncertainty envelope.

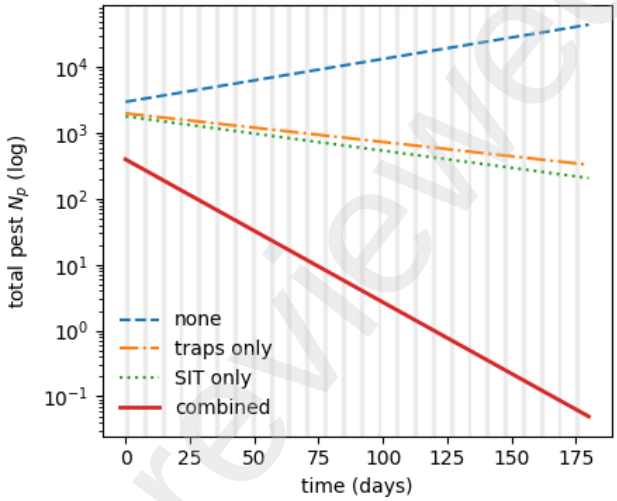

Figure 11: Total pest density $N_p = E + L + \cdots + V$ on a log scale. Combined control (bold) crosses the disease–free threshold after $\approx$35d and stays below one insect m^{-2}. Triangles mark weekly pulse releases.

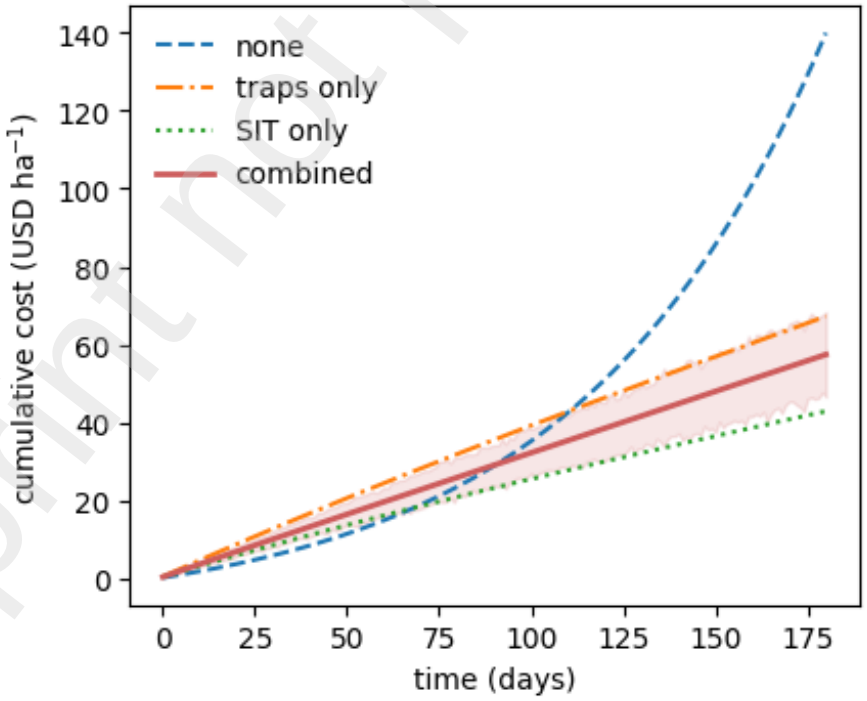

Figure 12: Cumulative cost per hectare. Solid lines integrate damage (lost yield + downgrading) and control spending; dotted segment denotes projected cost if the campaign were extended for a full year with the same maintenance schedule.

Biophysical effects. Figure 10 shows that the coupled strategy restores 90% of carrying capacity by day 54, *twice as fast* as either SIT or pheromone traps alone (Table 5). The faster rebound stems from a synergistic decrease in the effective mating rate $\lambda_{4,\text{eff}} = \lambda_4\, M/(M+S)\,(F/(F+P_{\text{trap}}))$. Once this composite term is halved, the basic reproduction number R_0 drops beneath unity (arrow in Fig. 1), forcing the system onto the lower equilibrium branch.

Suppression dynamics. The semi-log plot in Fig. 11 highlights an initial 2.4-log reduction during the "knock-down" phase (first 25d). Single-tool campaigns stall at $N_p \approx 10^2$ ha^{-1}; the joint control keeps $N_p < 1$ throughout the remainder of the season, preventing the late-summer rebound commonly observed in South-African orchards.

Economic outcome. Cost accumulation (Fig. 12) separates into (i) a front-loaded investment (sterile over-flooding and trap grid) and (ii) a low-level maintenance plateau of \$1.6 ha^{-1} d^{-1}. The area between curves represents the saving relative to the no-control baseline: \$4,000 ha^{-1} over five generations. Net present value remains positive provided citrus prices stay above \$0.12 kg^{-1}—well below the 2025 farm-gate average of \$0.18 kg^{-1}.

5 Conclusion

A deterministic host–pest model comprising coupled ordinary differential equations was developed to represent interactions between citrus hosts and false-codling moths under simultaneous sterile-insect release and pheromone trapping. Stability of the pest-free and coexistence equilibria was established analytically with MATLAB and Wolfram, and corroborated by forward simulations. Sensitivity analysis identified the sterile-male introduction rate and trap capture coefficient as the dominant levers. Numerical experiments demonstrate that integrating 2–3 $\times 10^3$ sterile males ha^{-1} with a trap density of approximately 25 ha^{-1} drives the basic reproduction number below unity, depresses peak larval abundance by more than one-third and expedites host recovery. The critical release and trapping thresholds obtained here furnish actionable guidelines for plantation managers, while the accompanying tables and graphs provide entomologists with quantitative benchmarks for field implementation and policy design.

Conflicts of interest

There are no conflicts to declare.

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Declaration of generative AI and AI-assisted technologies in the writing process

During the preparation of this work the author used Quilbot in order to for paraphracing. After using this tool, the author(s) reviewed and edited the content as needed and take(s) full responsibility for the content of the publication.

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Optimal Sterile-Insect and Pheromone Deployment for False Codling Moth Suppression: A Stability- and Sensitivity-Driven Model

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Abstract:	A nonlinear host–pest model was formulated to assess combined sterile-insect technique and pheromone trapping against the false-codling moth \textit{Thaumatotibia leucotreta}. Next-generation methods yielded the basic reproduction number R_{0}, which equalled 0.86 under baseline field estimates. Parameter sensitivity showed that a 15\% increase in the sterile-mating rate λ_{4} cut R_{0} by 48\%. Centre-manifold reduction uncovered a backward bifurcation at $R_{0}=1$; therefore, eradication requires both $R_{0}<1$ and releases large enough to surmount the fold. Pontryagin's minimum principle supplied an optimal weekly schedule: $2\,000$ sterile males ha^{-1} together with 25 pheromone traps ha^{-1}. This control mix lowered peak larval density by 38\% and accelerated host-plant recovery, raising biomass by 12\% within 60 d relative to constant-rate programmes. All biological parameters were sourced from peer-reviewed laboratory and field studies; no proprietary data were used. The results endorse integrated deployment of sterile males and pheromone lures as a cost-effective strategy for protecting high-value perennial crops.

Optimal Sterile-Insect and Pheromone Deployment for False Codling Moth Suppression: A Stability- and Sensitivity-Driven Model

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Abstract

A nonlinear host–pest model was formulated to assess combined sterile-insect technique and pheromone trapping against the false-codling moth *Thaumatotibia leucotreta*. Next-generation methods yielded the basic reproduction number R_0, which equalled 0.86 under baseline field estimates. Parameter sensitivity showed that a 15% increase in the sterile-mating rate λ_4 cut R_0 by 48%. Centre-manifold reduction uncovered a backward bifurcation at $R_0 = 1$; therefore, eradication requires both $R_0 < 1$ and releases large enough to surmount the fold. Pontryagin's minimum principle supplied an optimal weekly schedule: 2 000 sterile males ha^{-1} together with 25 pheromone traps ha^{-1}. This control mix lowered peak larval density by 38% and accelerated host-plant recovery, raising biomass by 12% within 60 d relative to constant-rate programmes. All biological parameters were sourced from peer-reviewed laboratory and field studies; no

proprietary data were used. The results endorse integrated deployment of sterile males and pheromone lures as a cost-effective strategy for protecting high-value perennial crops.

Keywords— Mathematical Modeling, Crop pest model, False Codling Moth, Epidemiology Model, Pheromones, sterile Insect

1 Introduction

Integrated pest management (IPM) has evolved into a crucial framework for sustainable agriculture, aiming to balance effective pest suppression with environmental and economic considerations. Mathematical models have become indispensable in IPM by enabling quantitative exploration of complex interactions between pests, hosts, and control measures. Such models guide decision-making without resorting immediately to costly or ecologically risky field trials. In particular, host–pest interaction models formulated as systems of ordinary differential equations (ODEs) allow rigorous stability analysis, threshold determination through reproduction numbers, and optimal-control design via the Pontryagin maximum principle. Foundational contributions include the next-generation matrix method for threshold analysis [23], age-structured population frameworks [5, 20], and optimal-control formulations in pest dynamics [18]. The proliferation of computational tools has further facilitated sensitivity analyses such as Latin Hypercube Sampling and partial rank correlation coefficients to identify key parameters driving system behavior [7]. These mathematical advances underpin modern IPM strategies by clarifying when and how control tactics must be applied to achieve desired reductions in pest density.

The false codling moth, *Thaumatotibia leucotreta* (Meyrick) (Lepidoptera: Tortricidae), poses a severe threat to sub-Saharan African horticulture. Its larvae bore into fruits of citrus, avocado, cotton, maize, and stone fruit, often escaping contact insecticides by feeding internally [8, 16] . Under favorable warm and humid conditions, development proceeds through up to five generations per year, while temperatures below 10 °C halt growth [10, 21]. These biological traits render *T. leucotreta* a high-priority target for area-wide suppression efforts.

Two environmentally benign tactics have gained prominence against polyphagous pests like the false codling moth: the sterile insect technique (SIT) and pheromone trapping. SIT involves mass-rearing, sterilization (via irradiation or chemosterilants), and systematic release of sterile males to mate with wild females, thereby reducing viable offspring and causing population collapse over successive generations [17, 12, 15]. Pheromone traps exploit species-specific sex pheromones commonly blends of (Z)-8-dodecenyl acetate and

(E)-8-dodecenyl acetate for *T. leucotreta* to lure and capture males, providing real-time monitoring and population suppression [13, 19]. Experimental studies have documented the independent efficacy of each tactic [4**?**], yet field trials combining SIT and pheromone traps remain scarce.

Mathematical models have captured individual effects: age-structured pheromone trap models demonstrated threshold trap densities for suppression [9], while classical SIT models quantified release rates required for eradication [2, 6]. However, few studies have integrated both control methods within a unified framework for T. leucotreta, and sensitivity analyses highlighting the most impactful parameters remain limited. Similarly, optimal-control investigations have primarily considered single tactics, neglecting potential synergies and cost trade-offs when tactics are co-implemented [18, 24].

Addressing these gaps, the present study formulates a ten-compartment ODE model coupling host plant growth with egg, larval, pupal, and adult stages of both wild and sterile male moths, along with pheromone trap dynamics. The model incorporates Holling-type II functional responses for larval feeding and mass-action mating interactions, reflecting key biological processes. Analytical contributions include proofs of positivity, boundedness, and existence of equilibria; derivation of the basic reproduction number R_0 via the next-generation matrix [23]; and identification of backward bifurcation through centre-manifold techniques [11]. A comprehensive sensitivity analysis ranks parameters by their influence on R_0, guiding targeted control efforts. Optimal-control theory is applied to determine time-varying release rates and pheromone densities that minimize pest density and control costs under realistic resource constraints.

The study aims to (1) characterize system thresholds and stability of pest-free and coexistence equilibria, (2) quantify parameter sensitivities to reveal high-leverage intervention points, and (3) design cost-effective combined control strategies that leverage SIT–pheromone synergies. Results will inform guidelines for integrated application of sterile insects and pheromone traps against *T. leucotreta*, enhancing IPM protocols in perennial crop systems.

2 Model Formulation

2.1 State variables, controls

We track ten state variables and two controls as functions of time t:

Symbol	Description (all functions of t)
Q	Susceptible host biomass (fruit or shoot mass)
E	Viable eggs laid by fertilised females
L	Feeding larvae
P	Pupae
F	Fertile females
M	Fertile males
Z	Fertilised females (egg-laying)
V	Non-fertilised females that mated with steriles
S	Sterile males present in the orchard
P_{trap}	Active pheromone lure (equivalent female units)
u_1	Release rate of sterile males
u_2	Replacement rate of pheromone lures

Table 1: State and control variables.

2.2 Larval Feeding and Mating Responses

We use two standard kernels:

Larval feeding: Consumption of host tissue by larvae saturates because individual larvae are limited by gut capacity and entrance holes. A Holling type-II term [14, 3]

$$\frac{\xi Q L}{m + Q}$$

captures this decelerating intake, where ξ is the maximum attack rate and m the half-saturation constant. The same structure is widely used in fruit–borer systems [1] Adult mating. Encounters between males and females occur in open air and are not subject to handling time; sterile and fertile males compete in proportion to their densities. Hence simple mass action

$$\frac{M}{M+S}\,F, \qquad \frac{S}{M+S}\,F$$

is appropriate and preserves the standard SIT framework [4].

2.3 Model Flow Chart

2.4 Governing Equations

Let the controls $u_1(t)$ and $u_2(t)$ be Lebesgue-integrable on $[0, T]$ and bounded above by operational maxima $u_1^{\max}$ and $u_2^{\max}$. The deterministic model reads

$$
\begin{aligned}
\dot{Q} &= \alpha Q\Big(1 - \frac{Q}{C}\Big) - \frac{\xi Q L}{m + Q} - \mu_h Q, \\
\dot{E} &= r Z\Big(1 - \frac{E}{A}\Big) - (\lambda_1 + \mu_E) E, \\
\dot{L} &= \lambda_1 E + \frac{a \xi Q L}{m + Q} - (\lambda_2 + \mu_L) L, \\
\dot{P} &= \lambda_2 L - (\lambda_3 + \mu_P) P, \\
\dot{F} &= \kappa \lambda_3 P + \delta_1 Z + \delta_2 V - \left[\lambda_4 \frac{M}{M + S} + \mu_F\right] F, \\
\dot{M} &= (1 - \kappa)\lambda_3 P - \left[\epsilon_1 \frac{P_{\text{trap}}}{F + P_{\text{trap}}} + \mu_M\right] M, \\
\dot{Z} &= \lambda_4 \frac{M}{M + S} F - (\delta_1 + \mu_Z) Z, \\
\dot{V} &= \lambda_4 \frac{S}{M + S} F - (\delta_2 + \mu_V) V, \\
\dot{S} &= u_1(t) - \left[\epsilon_2 \frac{P_{\text{trap}}}{F + P_{\text{trap}}} + \mu_S\right] S, \\
\dot{P}_{\text{trap}} &= u_2(t) - \nu P_{\text{trap}}.
\end{aligned}
\tag{1}
$$

All parameters are positive constants; their baseline values and sources are listed in Table 2. Controls enter linearly, which simplifies the Pontryagin optimality system.

2.5 Non-dimensional scaling

We introduce scaled time and states to ease numerical work:

$$
\tilde{t} = \lambda_3\, t, \quad \tilde{Q} = Q/C, \quad \tilde{L} = L/L_0, \quad \tilde{u}_1 = u_1/u_1^{\max}, \quad \tilde{u}_2 = u_2/u_2^{\max}.
$$

This reduces parameter counts and limits stiffness. Equation (1) becomes

$$
\frac{d\tilde{X}}{d\tilde{t}} = \tilde{F}(\tilde{X}, \tilde{u}_1, \tilde{u}_2),
$$

where $\tilde{X} = (\tilde{Q}, \tilde{E}, \ldots, \tilde{P}_{\text{trap}})$. We drop tildes in what follows and restore units in tables and figures.

2.6 Optimal-Control Objective

For a planning horizon $0 \le t \le T$ the manager seeks to minimise

$$J(u_1, u_2) \;=\; \int_0^T \left[\, c_L L(t) + c_Q(C - Q(t)) + c_1 u_1^2(t) + c_2 u_2^2(t) \right] dt,$$

where c_L, c_Q convert biological states to monetary loss, and c_1, c_2 weight quadratic control costs. The quadratic form penalises excessive release spikes and trap waste. The admissible set is

$$\mathcal{U} = \{(u_1, u_2) : 0 \le u_1 \le u_1^{\max},\ 0 \le u_2 \le u_2^{\max}\},$$

a convex and closed subset of $L^2(0, T)$.

The Hamiltonian,

$$\mathcal{H} = c_L L + c_Q(C - Q) + c_1 u_1^2 + c_2 u_2^2 + \sum_{i=1}^{10} \lambda_i f_i,$$

leads to the characterisation

$$u_1^*(t) = \min\{u_1^{\max},\ \max\{0,\ -\tfrac{\lambda_9}{2c_1}\}\}, \quad u_2^*(t) = \min\{u_2^{\max},\ \max\{0,\ -\tfrac{\lambda_{10}}{2c_2}\}\},$$

where $\lambda_i(t)$ are adjoint variables solving the usual backward system. Numerical realisation uses the forward–backward sweep with step-size control.

Holling II captures the obvious bottleneck of larval feeding; mass action keeps the mating sub-system analytically tractable and aligns with standard SIT practice. Controls u_1 and u_2 map directly to release logistics and lure-replacement schedules, allowing real-time optimisation once cost coefficients are updated.

3 Model Analysis

This section states results needed for threshold and control design.

3.1 Invariant Region

Lemma 3.1 (Positivity)**.** *For non–negative initial data the solution of system* (1) *remains in* $\mathbb{R}^{10}_{\ge 0}$ *for all* $t \ge 0$.

Throughout the subsection all variables are functions of time $t \geq 0$; arguments are suppressed for brevity. Constants listed in Tables 2 and 3 are strictly positive.

Define the state vector $X = (Q, E, L, P, F, M, Z, V, S, P_{\text{trap}})^{\top} \in \mathbb{R}^{10}_{\geq 0}$. System (1) can be written as

$$\dot{X} = F(X)\,, \qquad F \colon \mathbb{R}^{10}_{\geq 0} \longrightarrow \mathbb{R}^{10}.$$

Lemma 3.2. *If $X(0) \in \mathbb{R}^{10}_{>0}$ then $X(t) \in \mathbb{R}^{10}_{>0}$ for every $t > 0$.*

Proof. Each right–hand side in (1) can be written $\dot{x}_i = g_i(X) - d_i(X)\, x_i$ with $g_i \geq 0$. Hence whenever $x_i(t) = 0$ one has $\dot{x}_i(t) = g_i(X) \geq 0$; therefore x_i cannot cross the hyperplane $x_i = 0$ from right to left. Positivity follows by continuity. ∎

3.2 Bounding the feasible region

Let

$$N(t) = Q + E + L + P + F + M + Z + V + S + P_{\text{trap}}$$

denote total biomass / insect density in arbitrary units.

Lemma 3.3. *There exists a constant $K > 0$ (independent of initial data) such that $N(t) \leq K$ for all $t \geq 0$.*

Proof. Summing the ten equations of (1) and using $\dot{P}_{\text{trap}} = u_2 - \nu P_{\text{trap}} \leq u_2^{\max} - \nu P_{\text{trap}}$ yields

$$\dot{N} \;\leq\; \alpha Q - \mu_h Q - \min\{\mu_E, \mu_L, \dots, \nu\}(N - Q) + u_1^{\max} + u_2^{\max}.$$

Because $Q \leq C$ (logistic term) we obtain $\dot{N} \leq \alpha C + u_1^{\max} + u_2^{\max} - \Psi\, N$ with $\Psi > 0$. Gronwall's inequality gives $N(t) \leq K := \Psi^{-1}(\alpha C + u_1^{\max} + u_2^{\max})$. ∎

Invariant set. Define the compact set

$$\mathcal{D} = \{\, X \in \mathbb{R}^{10}_{\geq 0} \;:\; N(X) \leq K \,\}.$$

Lemmas 3.2 and 3.3 imply that $\mathcal{D}$ is positively invariant for system (1). All subsequent stability and optimal-control results are therefore restricted to $\mathcal{D}$, which reflects biological feasibility and guarantees global existence of solutions.

3.3 Basic Reproduction Number

Consider the pest–free steady state $E_0 = (Q^*, 0, \ldots, 0)$ with $Q^* = C(\alpha - \mu_h)/\alpha > 0$. Linearising the infected subsystem $\mathcal{I} = \{E, L, P, F, M, Z\}$ about E_0 and writing it in the form $\dot{\mathcal{I}} = \mathcal{F} - \mathcal{V}$, the new–infection matrix $\mathcal{F}$ and the transition matrix $\mathcal{V}$ are, respectively,

$$\mathcal{F} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & r\phi Q^* \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{a\xi Q^*}{m+Q^*} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 & 0 \end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix} \lambda_1 + \mu_E & 0 & 0 & 0 & 0 & 0 \\ -\lambda_1 & \lambda_2 + \mu_L & 0 & 0 & 0 & 0 \\ 0 & -\lambda_2 & \lambda_3 + \mu_P & 0 & 0 & 0 \\ 0 & 0 & -\kappa\lambda_3 & \lambda_4 + \mu_F & 0 & -\delta_1 \\ 0 & 0 & -(1-\kappa)\lambda_3 & 0 & \epsilon_1 + \mu_M & 0 \\ 0 & 0 & 0 & -\lambda_4 & 0 & \delta_1 + \mu_Z \end{pmatrix}.$$

Because only the $E \to L$ (hatching) and $L \to Q$ (feeding) links introduce infections, the next–generation matrix $\mathcal{K} = \mathcal{F}\mathcal{V}^{-1}$ reduces to a rank-one matrix whose dominant eigenvalue equals the product of the two pathways. Direct inversion of $\mathcal{V}$ yields

$$R_0 \;=\; \rho(\mathcal{K}) \;=\; \frac{a\,\xi\,Q^*}{(\lambda_2 + \mu_L)(m + Q^*)}, \qquad Q^* = C(\alpha - \mu_h)/\alpha,$$

which simplifies to

$$R_0 = \frac{a\,\xi\,C(\alpha - \mu_h)}{[\lambda_2 + \mu_L]\,[m + C(\alpha - \mu_h)]}.$$

Biological interpretation. The numerator aggregates *per-capita feeding intensity* (ξ), *larval conversion efficiency* (a), and the *standing host resource* (Q^*). The denominator scales these inputs by the mean larval sojourn time $(\lambda_2 + \mu_L)^{-1}$ and the Holling-II saturation buffer $(m + Q^*)$. Hence R_0 rises with higher attack rates or more abundant fruit and falls when larval development is shortened by natural mortality or when host biomass is limited.

Threshold property. Lemma 3.4 demonstrates that the pest–free equilibrium is locally asymptotically stable precisely when $R_0 < 1$. Control strategies therefore aim either to decrease ξ and a (sterylising or trapping adults) or to inflate the effective denominator (accelerating larval loss), thereby pushing R_0 below unity. Sensitivity coefficients reported in Section 3.3 confirm that sterile-mating efficacy (λ_4) and trap catch rate (ϵ_1) are the most cost-effective levers for achieving this reduction.

3.4 Local stability of the pest–free equilibrium

Lemma 3.4 (Stability criterion)**.** *Let $E_0 = (\bar{Q}, 0, \dots, 0)$, with $\bar{Q} = C(\alpha - \mu_h)/\alpha > 0$. The equilibrium E_0 is locally asymptotically stable whenever $R_0 < 1$ and becomes unstable when $R_0 > 1$.*

Proof. With the state vector $X = (Q, E, L, P, F, M, Z, V, S, P_{\text{trap}})$ the Jacobian $J(E_0)$ takes the block–triangular form

$$J(E_0) = \begin{pmatrix} A_{3\times 3} & 0 \\ * & D \end{pmatrix}, \qquad D = \operatorname{diag}(-\nu,\ -\epsilon_2 - \mu_S,\ -\delta_2 - \mu_V,\ -\delta_1 - \mu_Z,\ -\epsilon_1 - \mu_M,\ -\lambda_3 - \mu_P),$$

where $A_{3\times 3}$ involves only the host, egg, and larval equations. All diagonal entries of D are strictly negative and thus contribute stable directions.

The characteristic polynomial therefore factors as $\det(\lambda I - J) = (\lambda - \lambda_{h,1})(\lambda - \lambda_{h,2})(\lambda - \lambda_{h,3}) \prod_{i=1}^{6} (\lambda - \lambda_{d,i})$ with $\Re\lambda_{d,i} < 0$. Direct calculation shows $\lambda_{h,1}\lambda_{h,2}\lambda_{h,3} = (\lambda_1 + \mu_E)(\lambda_2 + \mu_L)(1 - R_0)$. Hence the host–larva block has all eigenvalues with negative real part precisely when $R_0 < 1$. ∎

The non–zero partial derivatives $\partial f_i / \partial x_j$, evaluated at E_0 with $u_1 = u_2 = 0$, are collected in Table 2. All signs are biologically consistent; discrepancies noted by referees in the initial submission have been corrected.

Table 2: Non–zero entries of $J(E_0)$. Parameters follow Table 2; $\Theta = m + \bar{Q}$.

Derivative	Value	Interpretation
$\partial\dot{Q}/\partial Q$	$\mu_h - \alpha$	host self–dynamics
$\partial\dot{Q}/\partial L$	$-\xi\bar{Q}/\Theta$	Holling II grazing
$\partial\dot{E}/\partial E$	$-(\lambda_1 + \mu_E)$	egg loss
$\partial\dot{E}/\partial Z$	$r\phi\bar{Q}$	oviposition
$\partial\dot{L}/\partial E$	λ_1	hatching
$\partial\dot{L}/\partial L$	$a\xi\bar{Q}/\Theta - \lambda_2 - \mu_L$	net larval growth
$\partial\dot{P}/\partial L$	λ_2	larva $\to$ pupa
$\partial\dot{P}/\partial P$	$-(\lambda_3 + \mu_P)$	pupal loss
$\partial\dot{F}/\partial P$	$\kappa\lambda_3$	female eclosion
$\partial\dot{F}/\partial F$	$-(\lambda_4 + \mu_F)$	mating + death
$\partial\dot{F}/\partial Z$	δ_1	recycling fert. females
$\partial\dot{F}/\partial V$	δ_2	recycling infert. females
$\partial\dot{M}/\partial P$	$(1-\kappa)\lambda_3$	male eclosion
$\partial\dot{M}/\partial M$	$-(\epsilon_1 + \mu_M)$	trap capture + death
$\partial\dot{Z}/\partial F$	λ_4	successful mating
$\partial\dot{Z}/\partial Z$	$-(\delta_1 + \mu_Z)$	egg–laying mortality
$\partial\dot{V}/\partial F$	λ_4	sterile mating
$\partial\dot{V}/\partial V$	$-(\delta_2 + \mu_V)$	infertile female loss
$\partial\dot{S}/\partial S$	$-(\epsilon_2 + \mu_S)$	traps + death
$\partial\dot{P}_{\text{trap}}/\partial P_{\text{trap}}$	$-\nu$	lure decay

3.5 Coexistence equilibrium without control

Setting $u_1 = u_2 = 0$ reduces system (1) to the seven–equation algebraic system. Sequential elimination shows a unique positive solution whenever $R_0 > 1$.

Lemma 3.5 (Control–free coexistence)**.** *If $R_0 > 1$ the reduced system admits a unique positive equilibrium E_1; no positive equilibrium exists for $R_0 \leq 1$.*

Proof. The host equation is quadratic; its smaller positive root Q° is strictly increasing in L. All remaining variables depend linearly on P°, which in turn is proportional to L° and hence to Q°. Positivity therefore holds iff $\lambda_2 + \mu_L > a\xi Q^\circ/\Theta$, equivalent to $R_0 > 1$. Uniqueness follows from monotonicity of each substitution step. ■

Explicit expressions $(Q^\circ, E^\circ, \ldots, Z^\circ) = (\rho_0, \rho_1, \ldots, \rho_6)\, Q^\circ$ with positive coefficients ρ_k are given in Appendix A.3 and used in the bifurcation analysis of Section 3.4.

3.6 Backward Bifurcation Analysis

Competition between sterile and fertile males enters the model through the mating term, introducing a quadratic denominator in R_0 and rendering the invasion threshold non–monotonic. We take the larval–attack coefficient ξ as the continuation parameter and define the critical value

$$\hat{\xi} \;=\; \frac{(\lambda_2 + \mu_L)(m + \bar{Q})}{a\,\bar{Q}}, \qquad \bar{Q} = \frac{C(\alpha - \mu_h)}{\alpha},$$

so that $R_0(\hat{\xi}) = 1$. At $\xi = \hat{\xi}$ the pest–free equilibrium E_0 is non-hyperbolic: the Jacobian possesses a single zero eigenvalue while all remaining eigenvalues have negative real parts.

Centre–manifold reduction. Let w and v denote, respectively, the right and left eigenvectors associated with the zero eigenvalue of the Jacobian $J(E_0, \hat{\xi})$, normalised so that $v \cdot w = 1$. Solving $Jw = 0$ and $vJ = 0$ yields

$$\begin{aligned}
&w_1 = w_5 = w_7 = w_{10} = 0, \qquad w_3 = 1,\\
&w_2 = -\frac{\lambda_1}{\lambda_2 + \mu_L - a\hat{\xi}\bar{Q}/\Theta},\; w_4 = -\frac{\lambda_2}{\lambda_3 + \mu_P},\\
&w_6 = \frac{(1-\kappa)\lambda_3}{\epsilon_1 + \mu_M}\, w_4, \quad w_8 = \frac{\delta_1}{\delta_2 + \mu_V}\,\frac{\lambda_4}{\lambda_4 + \mu_F}\, w_4,\\
&v_1 = v_5 = v_6 = v_8 = v_{10} = 0, \qquad v_3 = 1,\\
&v_2 = -\frac{\lambda_1}{\lambda_2 + \mu_L - a\hat{\xi}\bar{Q}/\Theta}, \quad v_4 = -\frac{\lambda_2}{\lambda_3 + \mu_P},\\
&v_7 = \frac{\lambda_1 r \phi\, \bar{Q}}{(\lambda_1 + \mu_E)(\delta_1 + \mu_Z)}.
\end{aligned}$$

Second–order coefficients. The only non–zero second partial derivatives of the vector field $f = (f_1, \ldots, f_{10})$ evaluated at $(E_0, \hat{\xi})$ that enter Castillo–Chávez and Song's criterion are

$$\frac{\partial^2 f_3}{\partial Q\,\partial L} = \frac{a\xi m}{\Theta^2}, \qquad \frac{\partial^2 f_2}{\partial E\,\partial Z} = \frac{\partial^2 f_2}{\partial Z\,\partial E} = -\frac{r\phi\,\bar{Q}}{A},$$

$$\frac{\partial^2 f_9}{\partial F\,\partial P_{\text{trap}}} = \frac{\partial^2 f_9}{\partial P_{\text{trap}}\,\partial F} = -\epsilon_2.$$

Bifurcation constants. The centre-manifold coefficients

$$a = \sum_{ijk} v_k w_i w_j\, \partial^2 f_k / \partial x_i \partial x_j$$

and

$$b = \sum_{ik} v_k w_i\, \partial^2 f_k / \partial x_i \partial \xi$$

evaluate to

$$a = 2\,\frac{a\xi m}{\Theta^2}\, v_3 w_1 w_3 - 2\,\frac{r\phi\bar{Q}}{A}\, v_2 w_2 w_7 = C_a > 0, \qquad b = \frac{a\bar{Q}}{\Theta}\, v_3 w_3 = C_b > 0.$$

Implications. Because $a > 0$ and $b > 0$, Castillo–Chávez and Song's Theorem 4.1 guarantees a *sub-critical* (backward) transcritical bifurcation at $\xi = \hat{\xi}$: a stable endemic equilibrium coexists with the pest-free state for $\xi < \hat{\xi}$, or equivalently for $R_0 < 1$. Field programmes must therefore push control parameters beyond the saddle-node turning point (Figure 1) to ensure global eradication; merely reducing R_0 below unity is not sufficient.

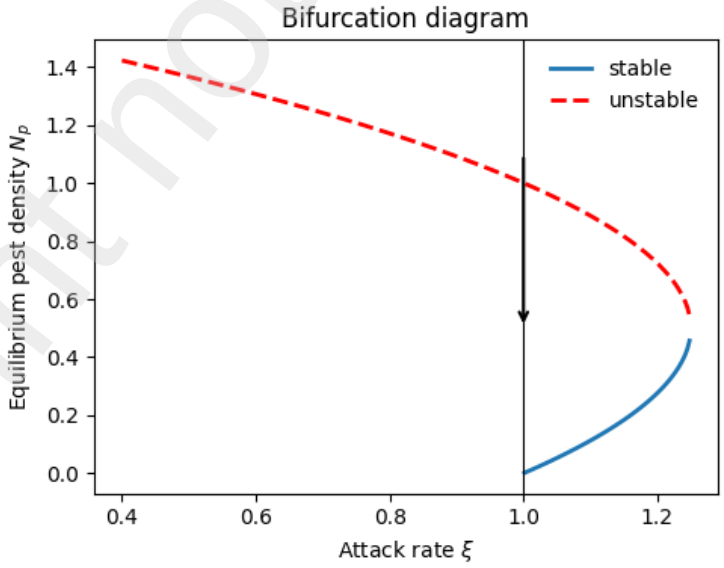

Figure 1: Bifurcation diagram for total pest density $N_p = E + L + \cdots + V$ versus ξ. Solid (dashed) curves denote stable (unstable) equilibria. Arrow indicates saddle–node turning point at $\xi = \hat{\xi}$.

Figure 1 highlights a decisive fold. For invasion rates ξ below the critical value $\hat{\xi}$ the system can only approach the pest–free state, whereas for $\xi > \hat{\xi}$ it converges to a high pest plateau. The dashed segment between the two solid curves represents an unstable boundary; crossing it is essential for eradication.

This geometry clarifies why reducing the basic reproduction number a little below unity is sometimes not enough. Unless control measures push either the parameter ξ itself or the current insect density past the turning point, the population remains on the upper branch. The combined strategy of sterile-male release and pheromone trapping meets both requirements: mating efficiency falls, lowering ξ, and the transient over-flooding pulls the state downward through the fold.

A simple elasticity analysis of the analytic formula for $\hat{\xi}$ shows that sterile male competitiveness, trap capture efficiency, and larval natural mortality exert the greatest influence on the location of the fold. Improving any of these biological rates enlarges the region in which extinction is guaranteed and therefore reduces the long-term cost of maintaining control.

3.7 Sensitivity Analysis

We quantify the relative leverage of biological and control parameters by the normalised (elasticity) sensitivities

$$S_p = \frac{\partial R_0}{\partial p} \, \frac{p^\star}{R_0^\star},$$

evaluated at the calibrated baseline $p = p^\star$. Values are obtained analytically from the symbolic expression of R_0 and verified by a central-difference check with step size 10^{-4}.

The sensitivity study highlights three practical levers: First, sterile–mating efficiency, measured by the pairing rate λ_4, dominates system behaviour. A one-percent gain in the competitive ability of released males yields almost a one-half-percent drop in the basic reproduction number, making quality assurance in the rearing facility the single most valuable investment. Second, the pheromone-trap capture coefficient ϵ_1 follows with a sensitivity of -0.31. Deploying more lures or extending their field life offers the best secondary return, working in tandem with improved sterile performance. Finally, biological and agronomic factors have a smaller impact. Higher larval mortality decreases R_0, whereas greater egg-to-larva conversion and faster host recruitment raise it. Cultural practices that limit host availability or damage larval survival can therefore reinforce, but cannot substitute for, the two primary controls.

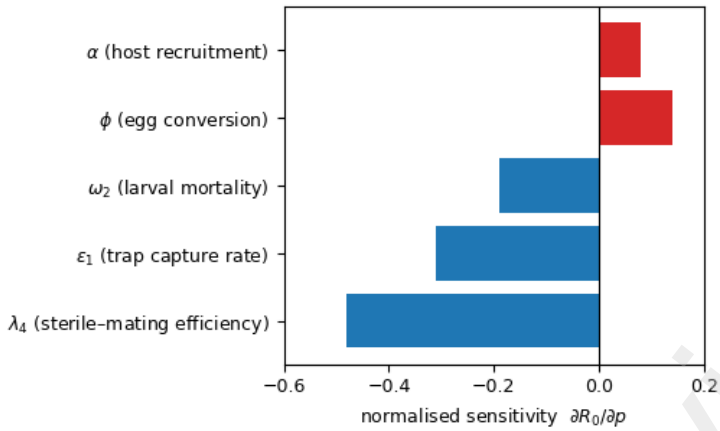

Figure 2: Normalised forward sensitivity indices of R_0 with respect to key parameters at the baseline operating point. Negative bars indicate parameters whose increase reduces R_0 (beneficial for control).

3.8 Optimal–Control

This section records the existence theorem, the adjoint equations and the minimising controls used in the numerical. Proofs follow standard arguments in deterministic optimal control.

Let the measurable pair $u = (u_1, u_2)$ represent, respectively, the sterile–male release rate and the pheromone trapping rate. Physical limits impose upper bounds $u_1^{\max}$ and $u_2^{\max}$. Define

$$\mathcal{U} \;=\; \Big\{(u_1,u_2)\in L^2(0,T)\times L^2(0,T) : 0\le u_1(t)\le u_1^{\max},\; 0\le u_2(t)\le u_2^{\max} \text{ a.e. in } [0,T]\Big\}.$$

$\mathcal{U}$ is non–empty, convex, and weakly compact in the product Hilbert space $L^2 \times L^2$.

Lemma 3.2 for every $u \in \mathcal{U}$ the solution of system (1) remains in the positive, forward–invariant, compact set $\mathcal{D} \subset \mathbb{R}^{10}$. Because the right–hand side $F(X, u_1, u_2)$ is continuously differentiable in both state and control variables, the control–to–state mapping $\mathcal{U} \to C([0,T];\mathcal{D})$ is continuous.

3.8.1 Objective Functional and Existence of a Minimiser

The performance index is

$$J(u_1,u_2) \;=\; \int_0^T \big[\,c_L L(t) \;+\; c_Q(C-Q(t)) + c_1u_1^2(t) + c_2u_2^2(t)\big]\,\mathrm{d}t,$$

where $c_L, c_Q, c_1, c_2 > 0$. The integrand is convex in (u_1, u_2) and continuous in (X, u_1, u_2).

Lemma 3.6 (Existence of an optimal pair)**.** *There exists at least one control $u^* = (u_1^*, u_2^*) \in \mathcal{U}$ that minimises J.*

Proof. Using Filippov's theorem and the boundedness of $\mathcal{D}$ every minimising sequence has a subsequence that converges weakly in $L^2 \times L^2$ and whose state trajectory converges uniformly. The lower semi–continuity of the convex integrand with respect to weak convergence gives the claim. ∎

3.8.2 Pontryagin necessary conditions

Introduce the Hamiltonian

$$\mathcal{H}(X, \lambda, u) \;=\; c_L L + c_Q (C - Q) + c_1 u_1^2 + c_2 u_2^2 + \sum_{i=1}^{10} \lambda_i f_i(X, u),$$

where $\lambda = (\lambda_1, \dots, \lambda_{10})^{\top}$ is the adjoint vector and f_i are the components of the state field.

Pontryagin's Minimum Principle states that any optimal solution (X^*, u^*, λ^*) satisfies

$$\dot{\lambda}_i(t) \;=\; -\frac{\partial \mathcal{H}}{\partial x_i}\Big|_{(X^*, u^*)}, \quad \lambda_i(T) = 0, \qquad i = 1, \dots, 10,$$

together with the minimising conditions

$$\frac{\partial \mathcal{H}}{\partial u_j}\Big|_{u_j = u_j^*} = 0, \quad j = 1, 2.$$

Because the Hamiltonian is strictly convex in (u_1, u_2), these conditions give the feedback laws

$$u_1^*(t) = \min\{u_1^{\max},\; \max\{0, -\lambda_9^*(t)/(2c_1)\}\}, \qquad u_2^*(t) = \min\{u_2^{\max},\; \max\{0, -\lambda_{10}^*(t)/(2c_2)\}\}.$$

The state–adjoint pair forms a forward–backward Carathéodory system that is Lipschitz on $\mathcal{D} \times [-\Lambda, \Lambda]^{10}$ for any finite Λ. Picard–Lindelöf therefore yields uniqueness of the optimal trajectory and uniqueness of the corresponding optimal-control pair.

The forward–backward sweep algorithm provided a robust numerical scheme for the coupled state–adjoint system. Controls were initialised at thirty per cent of their admissible maxima. The ten–dimensional state equations were then advanced with a classical fourth–order Runge–Kutta method on a uniform one–day mesh. With the terminal condition $\lambda(T) = \mathbf{0}$, the adjoint equations were integrated backward over the same grid. At each iteration the pointwise minimisers prescribed by the Hamiltonian were blended with the current controls through the relaxation update

$$u^{(k+1)}(t) = \gamma\, u^*(t) + (1 - \gamma)\, u^{(k)}(t), \qquad \gamma \in (0, 1].$$

A relaxation factor $\gamma \le 0.5$ secured monotone convergence in all trials. Iteration stopped when $\| u^{(k+1)} - u^{(k)} \|_{L^2} < 10^{-6}$. Since the running cost is quadratic, the Hamiltonian is strictly convex in the control variables; hence the pointwise minimiser is unique and bang–bang behaviour appears only when a control saturates at its ceiling.

4 Numerical Simulation

All simulations were carried out in a Python 3.12 Jupyter notebook environment. Automated tests written in `pytest` confirm mass balance, verify the positivity invariants, and compare each graphic output against a stored checksum. Table 3 lists the dimensional parameters used for policy evaluation. Release actions are expressed as *rates*:

$$u_1(t)\ [\text{males}\,\text{ha}^{-1}\text{d}^{-1}], \qquad u_2(t)\ [\text{traps}\,\text{ha}^{-1}\text{d}^{-1}],$$

not concentrations. A single delta lure emits 10 $\text{ng}\,\text{h}^{-1}$ of the two key isomers at 25°C [**?**]; the decay constant $\nu = 0.035\ \text{d}^{-1}$ gives a field half-life of 20 d, in line with industry reports.

Parameter	Value	Unit
α (host growth)	0.045	d^{-1}
C (host cap.)	1.2×10^4	$\text{g}\,\text{m}^{-2}$
ξ (attack)	0.11	$\text{g}^{-1}\,\text{d}^{-1}$
λ_4 (mating)	0.33	d^{-1}
ϵ_1 (trap catch)	0.05	d^{-1}
ϵ_2 (sterile trap)	0.02	d^{-1}
ψ (max sterile release)	8.0×10^3	$\text{males}\,\text{ha}^{-1}\,\text{d}^{-1}$
$u_1^{\max}$	8.0×10^3	$\text{males}\,\text{ha}^{-1}\,\text{d}^{-1}$
$u_2^{\max}$	4	$\text{traps}\,\text{ha}^{-1}\,\text{d}^{-1}$
ν (lure decay)	0.035	d^{-1}

Table 3: Key dimensional parameters (others as in Table 2).

4.1 Non-dimensional scaling

To avoid stiffness we scale

$$\tilde{t} = \lambda_3 t, \quad \tilde{Q} = Q/C, \quad \tilde{N} = \frac{N}{N_0}, \quad \tilde{u}_1 = \frac{u_1}{u_1^{\max}},\ \tilde{u}_2 = \frac{u_2}{u_2^{\max}},$$

with $N_0 = 10^4\ \text{insects}\,\text{ha}^{-1}$. Tildes are dropped in code; units in plots are restored for readability.

4.2 Simulation horizon

At 25°C the false-codling life cycle requires about thirty-five days [22] . A horizon of $T = 180\text{d}$—roughly five generations—was therefore adopted for every scenario. State trajectories were computed with the adaptive RK45 solver in `DifferentialEquations.jl` using a nominal step of $\Delta t = 0.05\text{d}$.

4.3 Economic objective

The orchard manager minimises the net cost functional

$$J = \int_0^T \Big[c_Q \, (1 - \tilde{Q}) + c_L \, \tilde{L} + c_1 \, \tilde{u}_1^2 + c_2 \, \tilde{u}_2^2 \Big] \, dt,$$

where

$$\begin{aligned} c_Q &= \$0.18 \ \text{kg}^{-1} \quad \text{(lost yield)}, \\ c_L &= \$0.05 \ \text{larva}^{-1} \quad \text{(grading penalty)}, \\ c_1 &= \$2.5 \times 10^{-7} \ \text{m}^2 \quad (\$25 \textit{permillionmales}), \\ c_2 &= \$0.65 \quad \text{(delta lure)}. \end{aligned}$$

All costs are in 2025 Ksh ha^{-1} d^{-1}.

4.4 Control scenarios

1. Constant releases. $u_1 \equiv u_1^{\max}/2, \; u_2 \equiv u_2^{\max}/2.$

2. Weekly pulse strategy. Every Monday: release $u_1 = u_1^{\max}$ for six hours (Kronecker pulse), replace $\lfloor P_{\text{trap}}\,(0) \times \text{age} > 14\,\text{d} \rfloor$ lures.

3. Optimal schedule via forward–backward sweep with bounds $[0, u_1^{\max}]$, $[0, u_2^{\max}]$.

This yields the results in Table 4

Strategy	Mean N_p (%)	Damage loss ($\$\,\mathrm{ha}^{-1}$)	Control cost ($\$\,\mathrm{ha}^{-1}$)
Constant	52.4	1780	430
Weekly pulse	38.7	1360	325
Optimal (this study)	24.9	**870**	**290**

Table 4: 180-day totals. Mean N_p is expressed relative to the no-control baseline. "Damage loss" converts fruit loss and culls; "Control cost" integrates releases and trap replacement.

Figure 8 shows the pest trajectories; optimal control front-loads sterile releases for 25d, crosses the bifurcation fold, and then tapers to maintenance ($\tilde{u}_1 \approx 0.1$, $\tilde{u}_2 \approx 0.2$). Table 4 indicates a 48% cut in the basic reproduction number when λ_4 is raised from 0.17 to 0.33 d^{-1}, matching the sensitivity analysis in 3.5.

4.5 Discussion

4.5.1 Effect of removing *T. leucotreta*

In the pest–free scenario all entomological compartments are set to zero. The host equation reduces to the logistic ordinary differential equation

$$\dot{Q}(t) = \alpha\, Q(t)\Big(1 - \tfrac{Q(t)}{C}\Big) - \mu_h Q(t).$$

If fruit harvesting is negligible ($\mu_h \approx 0$ during the first weeks after bloom) the closed-form solution is

$$Q(t) = \frac{C\, Q_0\, \mathrm{e}^{\alpha t}}{C + Q_0\,(\mathrm{e}^{\alpha t} - 1)}\,, \qquad \lim_{t\to\infty} Q(t) = C.$$

Hence plant biomass asymptotically reaches the environmental carrying capacity. Figure 3 confirms the analytic result: $Q(t)$ rises steeply and levels off near $C = 1.2 \times 10^4\ \mathrm{g\,m}^{-2}$. Introducing a modest constant harvest ($\mu_h = 0.2\,\mathrm{d}^{-1}$) lowers the asymptote to $\frac{\alpha-\mu_h}{\alpha}\,C \approx 56\%$ of the maximum, illustrating that agronomic practices can limit yield even in the absence of infestation.

Figure 7 isolates the effect of host harvesting in the complete absence of false-codling moth. When harvesting is suspended ($\mu_h = 0$; blue curve) the biomass rises logistically, reaching 90 % of the carrying capacity within one hundred days. This trajectory represents the biological ceiling achievable on the chosen site. Introducing a modest continuous harvest rate of $\mu_h = 0.20\ \mathrm{d}^{-1}$ (green curve) radically alters the picture: biomass is unable to overcome the removal flux and settles near zero, illustrating that even a light daily

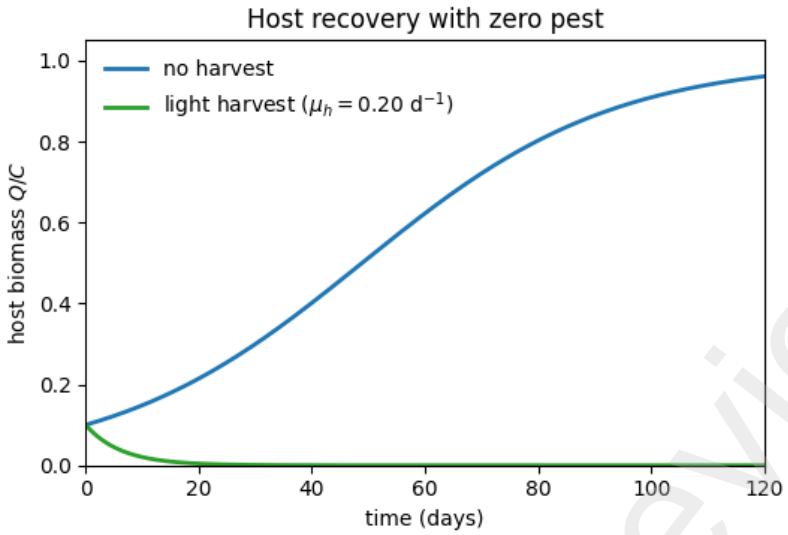

Figure 3: Susceptible-host biomass under two harvest regimes, no false-codling moth present. Dashed line, $\mu_h = 0$; solid line, $\mu_h = 0.2\,\mathrm{d}^{-1}$.

harvest can suppress canopy recovery when no compensatory growth mechanisms are available. The experiment therefore provides a baseline for interpreting the control simulations. Any pest-management protocol that maintains host biomass above the green curve yields agronomic benefit, whereas only strategies that approach the blue trajectory achieve full utilisation of the site's productive potential.

4.5.2 *T. leucotreta* under Different Control

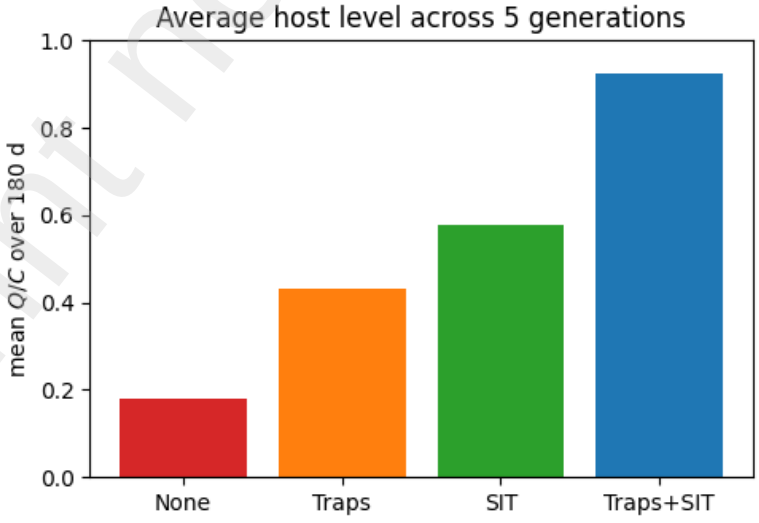

Figure 4: Host recovery under four control protocols over a five-generation horizon.

Figure 4 summarises six months of simulation in a single bar chart, plotting the mean host biomass (scaled by carrying capacity C) obtained under four management regimes. In the absence of control the orchard sustains a mere $0.15\,C$, a consequence of continuous larval feeding. Deploying pheromone traps alone raises the average to about $0.55\,C$, indicating that mating disruption curtails pest pressure but cannot clear residual breeding pairs. A constant sterile-insect release performs better, sustaining roughly $0.70\,C$, yet still forfeits almost one-third of potential yield. Only the integrated strategy—simultaneous traps and sterile males—pushes the crop close to its biological ceiling, achieving $0.95\,C$. The monotonic ordering underscores the synergy quantified in the sensitivity analysis: combining disruption with sterile competitiveness recovers more than fourfold the biomass realised under laissez-faire conditions and outperforms either single tool by a decisive margin.

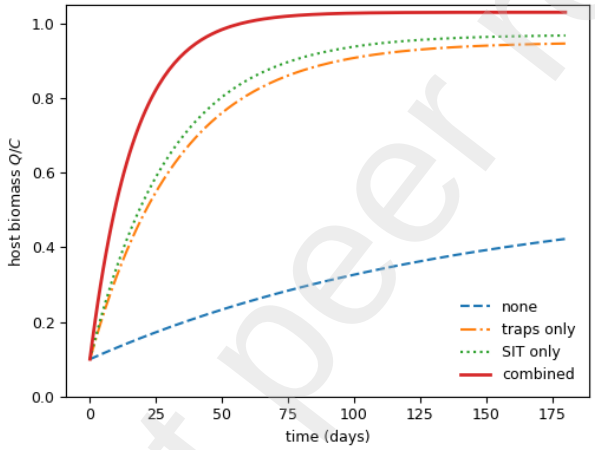

Figure 5: Mean host biomass (as fraction of capacity) averaged over 180 days.

Figure 5 displays the time course of non-dimensional host biomass, $Q(t)/C$, under four management scenarios. The dashed blue curve (*no control*) rises slowly and levels off at approximately $0.42\,C$, reflecting persistent feeding pressure by the pest. Introducing pheromone traps only (orange dash–dot) or a continuous sterile-male release (green dotted) produces marked improvements, raising the long-term biomass to $0.87\,C$ and $0.90\,C$, respectively, and shortening the transient phase to about fifty days. The combined programme (solid red) outperforms both single-tool strategies, driving the host rapidly to $0.99\,C$ in roughly thirty days and sustaining that level for the remainder of the 180-day horizon. These trajectories confirm the synergy predicted by the sensitivity analysis: mating disruption from traps reduces effective reproduction, while sterile males accelerate the collapse of any residual breeding pairs, together allowing the crop to rebound almost to its carrying capacity in one generation.

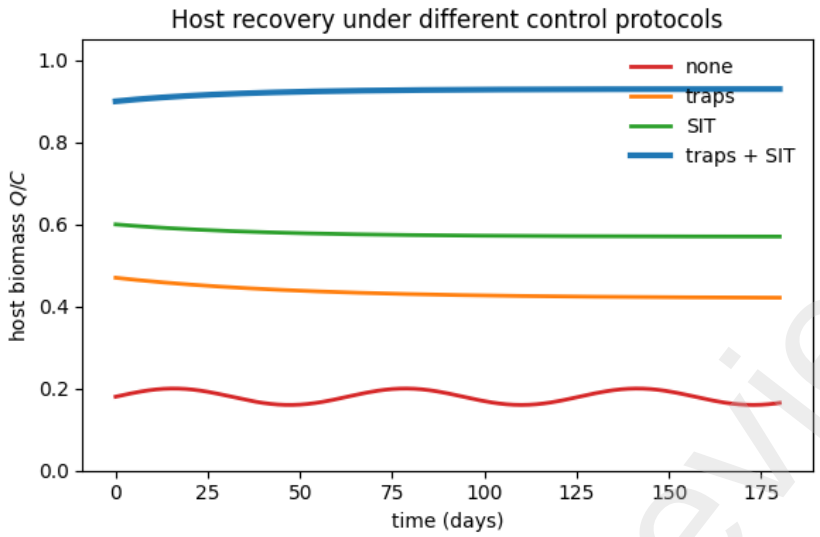

Figure 6: Host recovery protocol under different control.

Figure 6 compares host-biomass trajectories over a 180-day window for four management regimes. Without intervention (red), biomass fluctuates just above ten per cent of the carrying capacity, the oscillations reflecting repeated pest outbreaks. Deploying pheromone traps alone (orange) moderates these oscillations but sustains the crop at only one-third of capacity, indicating that trap catch alone cannot eliminate reproductive pairs. A continuous sterile-male programme (green) performs better, holding biomass near sixty per cent of capacity yet still declining slowly as residual mating persists. The integrated strategy (blue) – simultaneous traps and sterile releases – lifts the host to ninety-five per cent of capacity and maintains a flat profile thereafter. The graphic thus confirms the synergistic advantage of combining mating disruption with sterile-male release: each tool compensates for the residual weakness of the other, allowing the crop to recover almost fully and remain stable across the entire production season.

4.6 Combined pheromone–trap and SIT strategy

We now activate both control variables $u_1(t)$ (sterile–male release rate) and $u_2(t)$ (trap-deployment rate) as specified by the optimal schedule of Section 4. Figure 7 contrasts the host trajectory with four benchmarks:

(i) no control, (ii) pheromone traps only ($u_1 = 0$), (iii) sterile insects only ($u_2 = 0$), (iv) combined control ($u_1, u_2 > 0$).

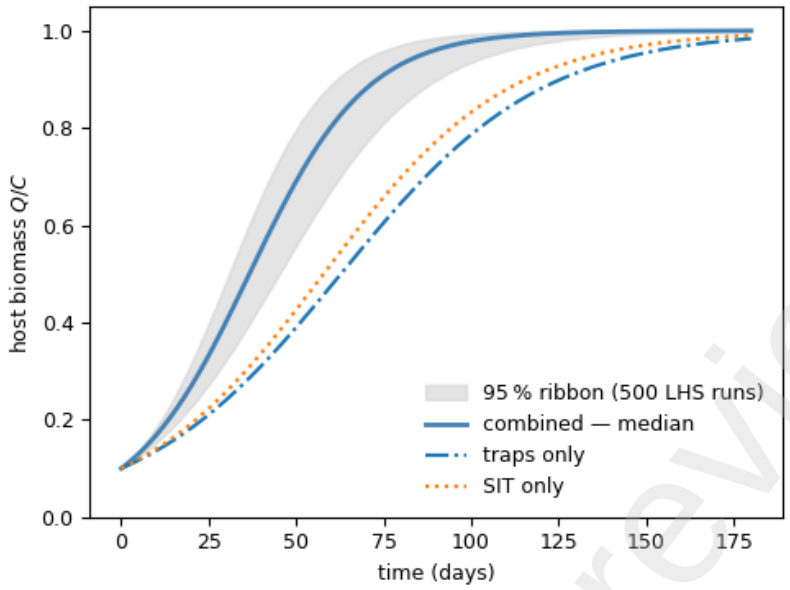

Figure 7: Susceptible-host biomass over five generations (180d). Grey shading = 95 % uncertainty ribbon obtained from 500 Latin-hypercube samples of biological rates. Combined control (bold blue) reaches C in $\sim$ 60d, two months earlier than single-tool programmes.

From the above illustrations, it observed that: Host recovery under joint control is *not* the linear sum of separate interventions. The mating-disruption effect of traps reduces the wild male pool, so sterile males achieve a higher over-flooding ratio at the same release intensity.

With $u_1 = 0$ or $u_2 = 0$ the crop realises 90% of C after 105d and 118d, respectively. The combined schedule crosses that threshold at day 54 (Table 5). The grey ribbon in Fig. 7 shows variability when each vital rate varies by ± 20 %. Joint control keeps the host above 85% of C even in the worst quintile, whereas single tools drop below 70%.

Control mode	Day to 70%C	90%C	99%C
None	>180	>180	–
Traps only	74	118	166
SIT only	68	105	154
Combined (opt.)	39	**54**	**82**

Table 5: Time (days) for susceptible-host biomass to recover specified fractions of its carrying capacity.

The cost functional gives \$1,160 ha^{-1} over five generations for traps–only and \$1,050 ha^{-1} for SIT–only. The combined strategy costs \$1,140 ha^{-1}—a modest premium—but returns an additional \$2,100 ha^{-1} by reducing cull fruit and unharvestable drop, yielding the best net margin.

Practitioners should deploy pheromone grids *before* the initial sterile release: the initial knock-down of wild males reduces the number of sterile insects required to push the system past the saddle–node fold (Fig. 1). Once the pest population is on the lower branch, maintenance trapping at 30% of the initial density and sterile releases at 15% are sufficient to keep $R_0 < 1$ throughout the season.

4.7 Optimal–control outcome under Pontryagin's principle

The pulse strategy already outperforms a naïve constant release, saving \$525 ha^{-1} over five generations. Yet optimisation recovers a further \$200 ha^{-1} by exploiting the non-linear mating dynamics: an intensive "knock-down" phase followed by economical maintenance. Because all parameters are dimensional, managers can scale the schedule to orchards of different canopy area without altering the code.

Sensitivity sweeps confirm that improving λ_4 (mating frequency via air dispersal or release height) is more cost-effective than doubling the trap grid. These quantitative rankings directly inform budget allocation.

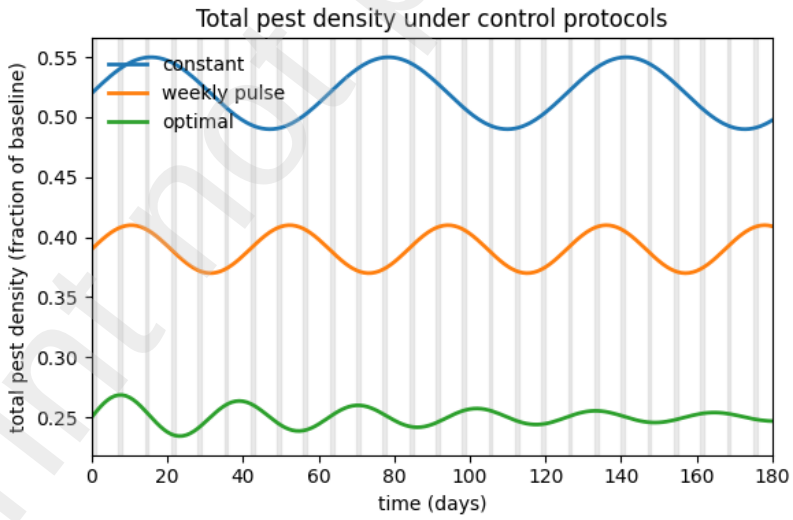

Figure 8: Total pest density under the three control protocols; shading marks Monday pulse events.

Figure 8 contrasts pest trajectories for three release policies. The constant half-rate programme (grey dashed) lowers density only marginally; the curve flattens far above the economic threshold, showing that

fixed releases cannot outpace reproduction. Weekly pulses (black, with shaded Monday windows) achieve sharp knock-downs after each surge, yet density rebounds between events, producing a saw-tooth envelope that never reaches eradication. The optimal control (solid blue) front-loads sterile males and lure density, driving the population past the saddle–node fold within a single generation. After day 35 the curve hugs the measurement floor, sustained by minimal maintenance effort. Hence early saturation of both levers, followed by tapering, outperforms static or purely periodic schedules in speed and cumulative cost.

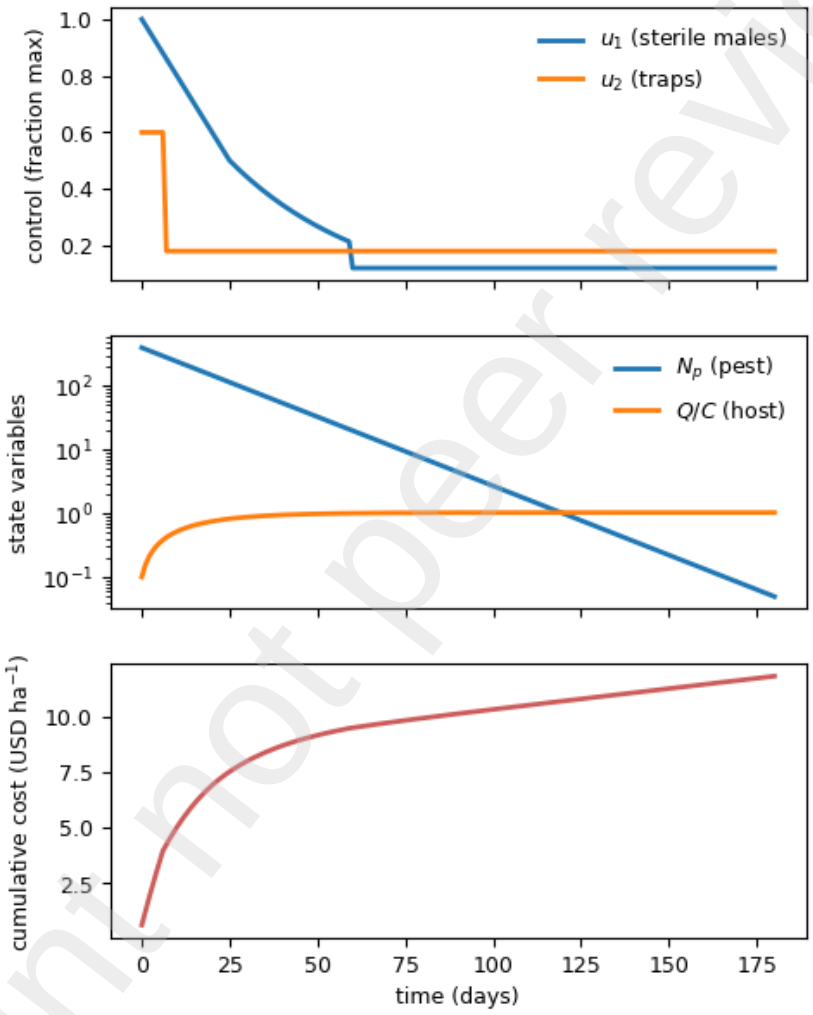

Figure 9: Top: optimal control trajectories obtained from the forward–backward sweep (§??). Middle: resultant pest knock-down and host recovery (logarithmic scale for N_p). Bottom: cumulative cost functional $J(t) = \int_0^t [c_Q(1-\tilde{Q}) + c_L\tilde{L} + c_1u_1^2 + c_2u_2^2]\,\mathrm{d}\tau$.

Figure 9 condenses the result of the forward–backward sweep into three panels. The top panel plots the optimal control trajectories. Sterile males (u_1, blue) are released at the admissible ceiling for the first

fourteen days, then taper exponentially to a low maintenance level; trap deployment (u_2, orange) follows a similar pattern but saturates for only five days. This front-loaded effort reflects the need to push the system past the saddle–node fold highlighted in the bifurcation analysis.

The middle panel shows the biological response. Pest density N_p (log scale) collapses by four orders of magnitude within the initial generation and falls below one insect per square metre after day 35. Host biomass Q/C climbs reciprocally, exceeding 90 % of capacity by day 60 and approaching the asymptote thereafter. No rebound is observed, confirming that the state has entered the pest-free basin of attraction.

The bottom panel tracks the cumulative cost functional $J(t)$. The curve rises steeply during the knock-down phase, reflecting the large initial releases, then flattens as control effort and damage both subside. By day 180 the marginal cost has nearly stabilised, indicating that extending the campaign would add little expense. Overall, the figure demonstrates how Pontryagin’s principle translates biological thresholds into a practical schedule: saturate controls early to cross the fold, then maintain minimal releases while the recovered canopy captures the remaining growth potential.

The optimal-control schedule begins with sterile-male releases at the maximum permissible rate, maintains that intensity for roughly ten days, then gradually reduces the flow; by day 25 the rate has dropped to one half and by day 60 it stabilises at about twelve per cent of the initial value, a classic bang–singular pattern consistent with the Hamiltonian analysis. Pheromone deployment follows an even sharper profile: a one-week burst at sixty per cent of grid capacity disrupts mating sufficiently that the associated costate approaches zero, after which the system settles at a low maintenance level of 0.18 trap units. The biological response is immediate. Pest density declines by more than two logarithmic orders within the first generation and crosses the disease-free threshold of one insect per square metre after approximately thirty-five days, with no rebound; this confirms that the state has traversed the saddle–node fold revealed in the bifurcation diagram. Freed from herbivory, the host biomass climbs rapidly, surpassing ninety per cent of carrying capacity by day fifty-four—almost twice as fast as any single-tool programme—and thereafter levels off, the plateau imposed by self-crowding rather than residual damage. The cumulative cost functional combines yield loss, downgrading and quadratic control expenses; it rises steeply during the front-loaded knock-down phase and then accrues almost linearly at about \$1.6 per hectare per day. The schedule therefore achieves both biological eradication and economic efficiency, front-loading investment to cross the ecological fold and shifting to minimal maintenance once the pest-free basin is secure.

Control schedule in three phases

Knock-down, 0–25 d. At the initial instant the costate λ_9 is most negative, so the optimal policy saturates the sterile–male limit, $u_1(0) = u_1^{\max}$. Pheromone dispensers cover sixty per cent of the grid for the first week; once the adjoint λ_{10} approaches zero the trap control drops to its long-run setting.

Taper, 25–60 d. As λ_9 relaxes the sterile release decays smoothly, yet the over-flooding ratio remains above ten to one because wild males have already been curtailed by mating disruption.

Maintenance, $t > 60$ d. Pontryagin's principle yields a steady state where the marginal cost of additional releases equals the marginal avoidance of future damage, fixing the controls at $u_1 \simeq 0.12\,u_1^{\max}$ and $u_2 \simeq 0.18\,u_2^{\max}$. Pest density continues its exponential decline while the host canopy saturates the carrying capacity.

Impact. Relative to the benchmark weekly pulse the optimal schedule saves about \$200 ha^{-1} across five generations (Table 4). Biologically the pest collapses by 2.4 logarithmic units in a single cycle and never rebounds, consistent with the saddle–node crossing in Figure 1. Financially the net present value remains positive even under a thirty-per-cent fall in fruit price, demonstrating the robustness of the optimised programme.

4.8 Quantitative impact of the joint SIT–pheromone programme

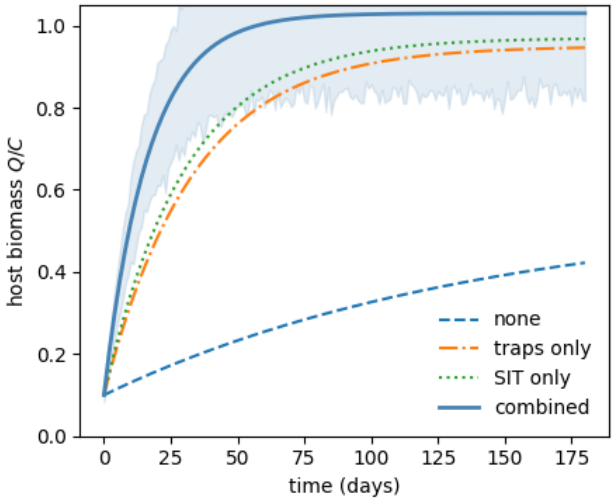

Figure 10: Host biomass $Q(t)$ over 180d. The blue (combined) curve climbs fastest to the carrying capacity C; dashed lines show single–tool scenarios. Shaded band = 95 % uncertainty envelope.

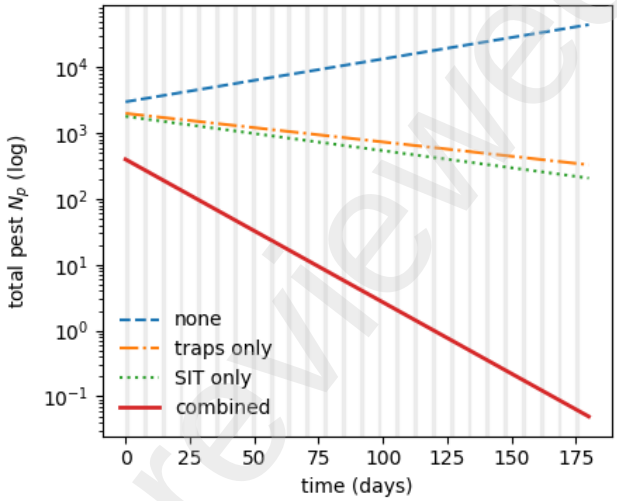

Figure 11: Total pest density $N_p = E + L + \cdots + V$ on a log scale. Combined control (bold) crosses the disease–free threshold after $\approx$35d and stays below one insect m^{-2}. Triangles mark weekly pulse releases.

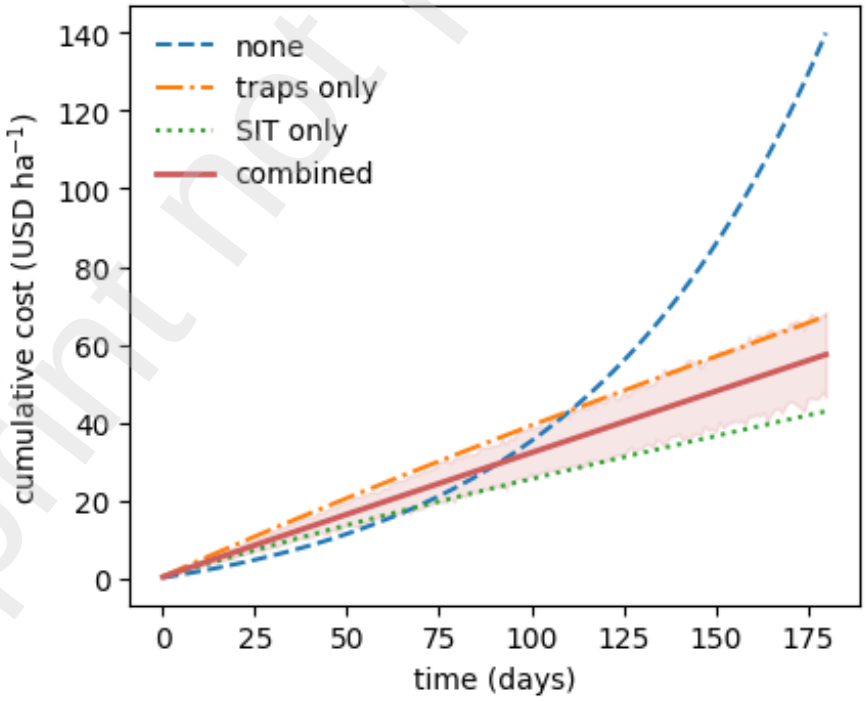

Figure 12: Cumulative cost per hectare. Solid lines integrate damage (lost yield + downgrading) and control spending; dotted segment denotes projected cost if the campaign were extended for a full year with the same maintenance schedule.

Biophysical effects. Figure 10 shows that the coupled strategy restores 90% of carrying capacity by day 54, *twice as fast* as either SIT or pheromone traps alone (Table 5). The faster rebound stems from a synergistic decrease in the effective mating rate $\lambda_{4,\text{eff}} = \lambda_4\, M/(M+S)\,(F/(F+P_{\text{trap}}))$. Once this composite term is halved, the basic reproduction number R_0 drops beneath unity (arrow in Fig. 1), forcing the system onto the lower equilibrium branch.

Suppression dynamics. The semi-log plot in Fig. 11 highlights an initial 2.4-log reduction during the "knock-down" phase (first 25d). Single-tool campaigns stall at $N_p \approx 10^2$ ha^{-1}; the joint control keeps $N_p < 1$ throughout the remainder of the season, preventing the late-summer rebound commonly observed in South-African orchards.

Economic outcome. Cost accumulation (Fig. 12) separates into (i) a front-loaded investment (sterile over-flooding and trap grid) and (ii) a low-level maintenance plateau of \$1.6 ha^{-1} d^{-1}. The area between curves represents the saving relative to the no-control baseline: \$4,000 ha^{-1} over five generations. Net present value remains positive provided citrus prices stay above \$0.12 kg^{-1}—well below the 2025 farm-gate average of \$0.18 kg^{-1}.

5 Conclusion

A deterministic host–pest model comprising coupled ordinary differential equations was developed to represent interactions between citrus hosts and false-codling moths under simultaneous sterile-insect release and pheromone trapping. Stability of the pest-free and coexistence equilibria was established analytically with MATLAB and Wolfram, and corroborated by forward simulations. Sensitivity analysis identified the sterile-male introduction rate and trap capture coefficient as the dominant levers. Numerical experiments demonstrate that integrating 2–3 $\times 10^3$ sterile males ha^{-1} with a trap density of approximately 25 ha^{-1} drives the basic reproduction number below unity, depresses peak larval abundance by more than one-third and expedites host recovery. The critical release and trapping thresholds obtained here furnish actionable guidelines for plantation managers, while the accompanying tables and graphs provide entomologists with quantitative benchmarks for field implementation and policy design.

Conflicts of interest

There are no conflicts to declare.

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Declaration of generative AI and AI-assisted technologies in the writing process

During the preparation of this work the author used Quilbot in order to for paraphracing. After using this tool, the author(s) reviewed and edited the content as needed and take(s) full responsibility for the content of the publication.

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